

# A formal category theoretic approach to the homotopy theory of dg categories

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# dg category theory

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(All will be considered over a field  $\mathbb{k}$  for simplicity.)

### Definition

A **differential graded category** (or **dg category**) is an enriched category over the monoidal category  $\text{Ch}(\mathbb{k})$  of complexes of  $\mathbb{k}$ -modules.

Precisely, a dg category  $\mathcal{A}$  consists of

- a collection of objects  $\text{ob}(\mathcal{A})$ ,
- Hom complexes  $\mathcal{A}(A, B)$  for  $A, B \in \mathcal{A}$ ,
- composition maps  $\circ: \mathcal{A}(B, C) \otimes_{\mathbb{k}} \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$  for  $A, B, C \in \mathcal{A}$ , and
- identities  $\text{id}_A \in Z^0(\mathcal{A}(A, A))$  for  $A \in \mathcal{A}$

such that these data satisfy the associativity and unity axioms.

## Typical example

The following form a dg category  $C_{\text{dg}}(\mathbb{k})$ :

- the objects are complexes of  $\mathbb{k}$ -modules

$$\dots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \rightarrow \dots$$

- the Homs are the complexes defined by the terms of degree  $n$

$$\text{Hom}^n(X, Y) = \prod_{i \in \mathbb{Z}} \text{Hom}(X^i, Y^{i+n})$$

with the differential given by

$$d^n(\{f^i\}_i) = \{d_Y^{i+n} \circ f^i - (-1)^n f^{i+1} \circ d_X^i\}_{i \in \mathbb{Z}}.$$

## Example

- Every preadditive category is a dg category with Hom cpxes concentrated in degree 0.
- A dg algebra  $A$  is exactly a dg category with one object.

## dg categories

From a dg category  $\mathcal{A}$ , we obtain:

- the **underlying category** (or  **$Z^0$ -category**)  $Z^0(\mathcal{A})$  by defining  $Z^0(\mathcal{A})(A, B) = Z^0(\mathcal{A}(A, B))$ ;
- the **homotopy category** (or  **$H^0$ -category**)  $H^0(\mathcal{A})$  by defining  $H^0(\mathcal{A})(A, B) = H^0(\mathcal{A}(A, B))$ .

Also, we define:

- A **(right) dg  $\mathcal{A}$ -module** := a dg functor  $\mathcal{A}^{\text{op}} \rightarrow C_{\text{dg}}(\mathbb{k})$ .
- The category of dg modules  $C(\mathcal{A}) := \text{dgFun}(\mathcal{A}^{\text{op}}, C_{\text{dg}}(\mathbb{k}))$ .
- The **derived category**  $D(\mathcal{A})$  of  $\mathcal{A}$  := the localization of  $C(\mathcal{A})$  w.r.t. quasi-isomorphisms.
- Note:  $D(\mathcal{A})$  is a triangulated category.

Every  $H^0$ -category has the canonical Yoneda-like embedding

$$H^0(\mathcal{A}) \hookrightarrow D(\mathcal{A}), \quad A \mapsto \mathcal{A}(-, A)$$

into the derived category.

## dg categories

Dg categories play an important role in algebraic geometry and representation theory as an enhanced notion of triangulated categories.

A triangulated category is an additive category equipped with a shift functor and a collection of distinguished triangles satisfying some axioms. In particular, every morphism admits a cone and a cocone, though the construction is not functorial.

### Definition

A dg category  $\mathcal{A}$  is **pretriangulated** if  $H^0(\mathcal{A})$  is closed (up to iso.) under shifts and cones of the derived category  $D(\mathcal{A})$  via the embedding  $H^0(\mathcal{A}) \hookrightarrow D(\mathcal{A})$ .

If  $\mathcal{A}$  is pretriangulated, then  $H^0(\mathcal{A})$  is a triangulated category (a triangulated full subcategory of  $D(\mathcal{A})$ ).

$\rightsquigarrow$  Hence, (pretriangulated) dg categories are considered as **enhanced triangulated categories**. [Bondal–Kapranov 1991]

# The homotopy theory of dg categories

Dg categories are not merely enriched categories, but also carry a homotopy-theoretic structure.

## Definition

A dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a **quasi-equivalence** if

- all  $F_{A,A'}: \mathcal{A}(A, A') \rightarrow \mathcal{B}(FA, FA')$  are quasi-isomorphisms,
- $H^0(F): H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$  is essentially surjective.

## Theorem (Tabuada 2005)

The category **dgCat** of (small) dg categories admits a model category structure whose weak equivalences are quasi-equivalences.

We refer to the localization of **dgCat** w.r.t. quasi-equivalences

$$\text{dgCat} \longrightarrow \text{Ho}(\text{dgCat})$$

as **the homotopy category of dg categories**.

# Formal category theory

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# Formal category theory

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**Formal category theory** (FCT) is an approach that seeks to formalize “category theory” itself in the 2-categorical setting.

- Just as **abstract homological algebra** formalizes classical arguments in  $\text{Mod}(R)$ , **formal category theory** formalizes arguments in **the 2-category  $\text{Cat}$  of categories**.

One of the frameworks for FCT is a **proarrow equipment**.

- **Abelian/triangulated categories** are to **homological algebra** what **proarrow equipments** are to **formal category theory**.
- With the structure of a proarrow equipment, one can interpret many concepts in category theory, including **(co)limits**, in an abstract way.

# Proarrow equipments

Let  $\mathcal{K}$  be a bicategory (=weak 2-category).

## Definition (Wood 1982, 1985)

A **proarrow equipment** on  $\mathcal{K}$  is a pseudofunctor  $(-)_* : \mathcal{K} \rightarrow \mathcal{M}$  to a bicategory  $\mathcal{M}$  which satisfies the following.

- (1)  $(-)_*$  is bijective on objects.
- (2)  $(-)_*$  is locally fully faithful (= bij. on 2-cells).
- (3) For any morphism  $f$  of  $\mathcal{K}$ ,  $f_*$  has a right adjoint  $f^*$  in  $\mathcal{M}$ .

## Example

Let  $\mathcal{K} = \text{Cat}$ . Consider  $\mathcal{M} = \text{Prof}$ , the bicategory of **profunctors**. A **profunctor**  $\mathcal{A} \rightarrow \mathcal{B}$  is just a Set-valued functor  $\mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}$ . We define a pseudofunctor  $(-)_* : \text{Cat} \rightarrow \text{Prof}$  by

$$(F : \mathcal{A} \rightarrow \mathcal{B}) \mapsto (F_* = \text{Hom}_{\mathcal{B}}(-, F-)) : \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \text{Set}.$$

Then  $F_*$  has a right adjoint  $F^* = \text{Hom}_{\mathcal{B}}(F-, -)$ , and hence  $(-)_* : \text{Cat} \rightarrow \text{Prof}$  is a proarrow equipment.

### Example

Similarly, let  $\mathcal{K} = \text{dgCat}$ , and  $\mathcal{M} = \text{Bimod}$ , the bicategory of **dg bimodules**.

Here a **dg bimodule**  $\mathcal{A} \leftrightarrow \mathcal{B}$  is a dg-functor  $\mathcal{B}^{\text{op}} \otimes \mathcal{A} \rightarrow C_{\text{dg}}(\mathbb{k})$ . Then the pseudofunctor  $(-)_* : \text{dgCat} \rightarrow \text{Bimod}$ , which sends

$$(F : \mathcal{A} \rightarrow \mathcal{B}) \mapsto (F_* = \mathcal{B}(-, F-): \mathcal{B}^{\text{op}} \otimes \mathcal{A} \rightarrow C_{\text{dg}}(\mathbb{k})),$$

becomes a proarrow equipment.

More generally, for a suitable monoidal category  $\mathcal{V}$ , we have a proarrow equipment  $(-)_* : \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Prof}$  of  $\mathcal{V}$ -enriched categories.

# The notion of limits in a proarrow equipment

Let  $(-)_*: \mathcal{K} \rightarrow \mathcal{M}$  be a proarrow equipment.

## Definition (Wood 1982)

Let  $f: J \rightarrow A$  be a morphism of  $\mathcal{K}$  and  $W: J \rightarrow \mathcal{M}$  be a morphism of  $\mathcal{M}$ . The  **$W$ -weighted limit** of  $f$  is a pair of

- a morphism  $\lim^W f: M \rightarrow A$  in  $\mathcal{K}$ ,
- a right Kan extension in  $\mathcal{M}$

$$\begin{array}{ccc} M & & \\ W \uparrow & \searrow^{(\lim^W f)_*} & \\ J & \xrightarrow{f_*} & A. \\ & \pi \Downarrow & \end{array}$$

In other words, the limit  $\lim^W f$  exists

$\iff$  the right Kan extension  $\text{Ran}_W f_*$  in  $\mathcal{M}$  exists and is representable (by a morphism of  $\mathcal{K}$ ).

We call  $f$  the **diagram** of the limit and  $W$  the **weight**.

## Example of limits

### Example

In the proarrow equipment  $(-)_* : \text{Cat} \rightarrow \text{Prof}$ , let

- $F : \mathcal{J} \rightarrow \mathcal{A}$  be a functor;
- $W = \Delta_{\{*\}} : \mathcal{J} \rightarrow \mathbf{1}$  be the const functor  $\mathcal{J}^{\text{op}} \rightarrow \text{Set}$  at  $\{*\}$ .

Then  $\text{Ran}_W F_* : \mathbf{1} \rightarrow \mathcal{A}$  is the functor  $\text{Cone}(-, F)$  of cones over  $F$ . Hence, the limit is equal to the **ordinary limit** of  $F$ .

### Example (dg terminal object)

In the proarrow equipment  $(-)_* : \text{dgCat} \rightarrow \text{Bimod}$ , let

- $\mathcal{J} = \emptyset$ , the empty dg category;
- $F : \emptyset \rightarrow \mathcal{A}$  be the unique functor;
- $W = 0 : \emptyset \rightarrow \mathbb{k}$  be the unique module.

Then  $\text{Ran}_W F_*$  is the zero module (the terminal object). Hence, the limit is equal to the **dg terminal object** in  $\mathcal{A}$ : i.e., an object  $A_0 \in \mathcal{A}$  such that  $\mathcal{A}(A, A_0) \cong 0$  for all  $A$ .

# FCT meets homotopy theory

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# FCT meets homotopy theory

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We have the proarrow equipment of dg categories

$$(-)_* : \text{dgCat} \longrightarrow \text{Bimod},$$

which provides enriched category theory of dg categories.

However, this equipment clearly does not take into account the homotopy theory of dg categories.

Can we obtain a more appropriate proarrow equipment for dg categories?

## dg bimodules

Recall that **Bimod** is the bicategory with Hom-categories  $C(\mathcal{A}, \mathcal{B}) = \text{Fun}(\mathcal{B}^{\text{op}} \otimes \mathcal{A}, C_{\text{dg}}(\mathbb{k}))$ .

### Proposition

Taking the derived category of each  $C(\mathcal{A}, \mathcal{B})$ , we obtain a new bicategory **DBimod** with composition given by the derived tensor product  $\otimes^{\mathbb{L}}$  of bimodules (this is associative if  $\mathbb{k}$  is a field).

In **Bimod**, a dg bimodule  $X: \mathcal{A} \rightarrow \mathcal{B}$  is written as the form of  $F_*$  for some dg functor  $F$  iff it is **right representable**, i.e., for all  $A$   $X(-, A): \mathcal{B}^{\text{op}} \rightarrow C_{\text{dg}}(\mathbb{k})$  is isomorphic to a representable  $\mathcal{B}$ -mod.

### Definition

A dg bimodule  $X: \mathcal{A} \rightarrow \mathcal{B}$  is **right quasi-representable** iff for all  $A \in \mathcal{A}$ ,  $X(-, A)$  is quasi-isomorphic to a representable  $\mathcal{B}$ -module.

- $D(\mathcal{A}, \mathcal{B})^{\text{rqr}} \subseteq D(\mathcal{A}, \mathcal{B})$ : the full subcategory of rqr bimods.
- **DBimod**<sup>rqr</sup>  $\subseteq$  DBimod: the sub-bicategory of rqr bimods.

# Quasi-functors

Rqr dg bimodules are considered as a kind of *weak dg functors* and are also called **quasi-functors**. In fact, the following is known.

## Proposition (Toën 2007)

There exists a bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dgCat})}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Iso}(D(\mathcal{A}, \mathcal{B})^{\mathrm{rqr}}).$$

The next proposition supports that  $\mathrm{DBimod}^{\mathrm{rqr}}$  would work as a 2-categorical refinement of  $\mathrm{Ho}(\mathrm{dgCat})$ .

## Proposition

A dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a quasi-equivalence

$\iff$  the associated quasi-functor  $F_* = \mathcal{B}(-, F(-)): \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence in  $\mathrm{DBimod}^{\mathrm{rqr}}$ .

# Proarrow equipment for homotopy theory

The following is a homotopical analogue of the fact that dg functors have right adjoints in  $\text{Bimod}$ .

## Proposition (Genovese 2017)

An  $\text{rqr}$  dg bimodule (quasi-functor) has a right adjoint in  $\text{DBimod}$ .

In summary, we have:

## Theorem (I.)

The inclusion  $\text{DBimod}^{\text{rqr}} \hookrightarrow \text{DBimod}$ , denoted by  $(-)_*$ , is a proarrow equipment on  $\text{DBimod}^{\text{rqr}}$ .

Applying the notion of limits to this proarrow equipment, we obtain the notion of **homotopy limits** in a dg category.

# Homotopy limits in a dg category

An **h-terminal object** of  $\mathcal{A}$  can be defined as the  $W$ -weighted limit of  $f$  for suitable  $f$  and  $W$  in the proarrow equipment  $(-)_\star: \text{DBimod}^{\text{rqr}} \rightarrow \text{DBimod}$ .

## Proposition (I.)

A dg category  $\mathcal{A}$  has an h-terminal object  $\iff$  it has an object  $A_0$  such that  $\mathcal{A}(A, A_0) \simeq_{\text{qism}} 0$  for all  $A$ .

- Note:  $A_0$  is dg terminal iff  $\mathcal{A}(A, A_0) \cong 0$  for all  $A$ .

From this, it appears that  $(-)_\star$  captures the homotopy theory.

## Homotopy limits in a dg category

We can also define **h-products**, **h-shifts**, and **h-cocones** as homotopy limits for suitable diagrams and weights.

### Proposition (I.)

A dg category  $\mathcal{A}$  has

- (1) all h-products  $\iff H^0(\mathcal{A})$  has products and the embedding  $H^0(\mathcal{A}) \hookrightarrow D(\mathcal{A})$  preserves them.
- (2) all h-shifts  $\iff H^0(\mathcal{A})$  is closed under shifts of  $D(\mathcal{A})$ .
- (3) all h-cocones  $\iff H^0(\mathcal{A})$  is closed under cocones of  $D(\mathcal{A})$ .

Thus we have a formal characterization of pretriangulatedness.

### Corollary (I.)

A dg category  $\mathcal{A}$  is pretriangulated

$\iff$  it has all h-shifts and h-cocones.

## Some properties and remarks

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Dually, we have h-coproducts, h-coshifts, and h-cones as the corresponding homotopy colimits.

- $\mathcal{A}$  has h-coproducts  $\iff H^0(\mathcal{A})$  has coproducts and the embedding  $H^0(\mathcal{A})^{\text{op}} \hookrightarrow D(\mathcal{A}^{\text{op}})$  preserves them.

All homotopy colimits can be constructed from these, as follows.

## Theorem (I.)

A dg category  $\mathcal{A}$  admits all homotopy colimits

$\iff$  it has h-coproducts, h-coshifts, and h-cones.

This theorem is essentially a consequence of the compact generation of derived categories.

In a general proarrow equipment, a (co)limit is called **absolute** if it is preserved by any arrow.

## Proposition

Limits weighted by right adjoints are absolute.  
Dually, colimits weighted by left adjoints are absolute.

## Proposition

Let  $f$  be a diagram and  $W \dashv V$  be an adjunction of weights.  
Then, an arrow  $z$  is a  $W$ -weighted colimit of  $f$  if and only if  $z$  is a  $V$ -weighted limit of  $f$ ; and these are both absolute.

Returning to the equipment  $(-)_*$ , we have the following characterization of adjoint weights.

## Theorem (I.)

For a dg bimodule  $X : \mathcal{A} \rightarrow \mathcal{B}$ , TFAE:

- (1)  $X$  has a right adjoint in  $\text{DBimod}$ .
- (2)  $X$  is right compact; i.e.  $X(-, A) \in D(\mathcal{B})$  is compact for each  $A \in \mathcal{A}$ .<sup>1</sup>

## Example

**H-terminal objects**, **h-shifts**, and **h-cocones** are weighted by right adjoints and hence absolute limits. They are equivalent to **h-initial objects**, **h-coshifts**, and **h-cones**, respectively.

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<sup>1</sup>An object  $C$  of a triangulated category  $\mathcal{T}$  with coproducts is called **compact** if  $\text{Hom}_{\mathcal{T}}(C, -) : \mathcal{T} \rightarrow \text{Ab}$  preserves coproducts.

# Cauchy completeness

## Proposition

For an object  $A$  in a general proarrow equipment, TFAE:

- (1)  $A$  is **Cauchy complete**: i.e., all left adjoint proarrows with codomain  $A$  are representable by arrows.
- (2)  $A$  has all limits weighted by right adjoints.
- (3)  $A$  has all colimits weighted by left adjoints.

It follows that every object with all absolute limits is Cauchy complete.

## Proposition (I.)

For a dg category  $\mathcal{A}$ , TFAE:

- (1)  $\mathcal{A}$  is h-Cauchy complete.
- (2) The inclusion  $H^0(\mathcal{A}) \hookrightarrow \text{Perf}(\mathcal{A}) := D(\mathcal{A})^{\text{cpt}}$  is an equivalence of categories.

Here  $D(\mathcal{A})^{\text{cpt}}$  denotes the full subcategory of compact objects.

# Adjunctions of quasi-functors

## Definition (Genovese 2017)

An **adjunction of quasi-functors** refers to an adjunction in the bicategory  $\text{DBimod}^{\text{rqr}}$  of quasi-functors.

Adjunctions of quasi-functors relate to other adjunctions, as follows.

## Proposition (I.)

- (1) For an adjunction  $F \dashv G: \mathcal{A} \rightarrow \mathcal{B}$  of dg functors, we have an adjunction  $F_* \dashv G_*$  of quasi-functors.
- (2) Any adjunction  $f \dashv g: \mathcal{A} \rightarrow \mathcal{B}$  of quasi-functors induces an adjunction  $H^0(f) \dashv H^0(g): H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ .

Proof: Because there are pseudofunctors  $\text{dgCat} \rightarrow \text{DBimod}^{\text{rqr}}$  and  $\text{DBimod}^{\text{rqr}} \rightarrow \text{Cat}$  (discuss later).

## Reflection of colimits and adjoints

If  $\mathcal{A}$  has h-coproducts, then  $H^0(\mathcal{A})$  has coproducts in the usual sense. The converse does not hold in general, but we have:

### Theorem (I.)

Suppose  $\mathcal{A}$  has h-shifts.

Then,  $H^0(\mathcal{A})$  admits coproducts  $\implies \mathcal{A}$  has h-coproducts.

### Corollary

A pretriangulated dg category  $\mathcal{A}$  admits all homotopy colimits  $\iff$  the triangulated category  $H^0(\mathcal{A})$  has coproducts.

Using the same idea, we also obtain:

### Theorem (I.)

Let  $f: \mathcal{A} \rightarrow \mathcal{B}$  be a quasi-functor where  $\mathcal{A}, \mathcal{B}$  have h-shifts. Then, the induced functor  $H^0(f): H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$  has a right adjoint  $\implies f$  has a right adjoint as a quasi-functor.

## A universal property

- The proarrow equipment  $(-)_* : \mathbf{dgCat} \hookrightarrow \mathbf{Bimod}$  is for (strict) category theory of dg categories,
- The proarrow equipment  $(-)_* : \mathbf{DBimod}^{\text{rqr}} \hookrightarrow \mathbf{DBimod}$  is for homotopy category theory of dg categories.

### Proposition (I.)

There are a lax functor  $\Gamma : \mathbf{Bimod} \rightarrow \mathbf{DBimod}$  and a pseudofunctor  $\gamma : \mathbf{dgCat} \rightarrow \mathbf{DBimod}^{\text{rqr}}$  such that  $\Gamma \circ (-)_* = (-)_* \circ \gamma$ .

$$\begin{array}{ccc} \mathbf{dgCat} & \xleftarrow{(-)_*} & \mathbf{Bimod} \\ \gamma \downarrow & & \downarrow \Gamma \\ \mathbf{DBimod}^{\text{rqr}} & \xleftarrow{(-)_*} & \mathbf{DBimod} \end{array}$$

### Proposition (I.)

The lax functor  $\Gamma : \mathbf{Bimod} \rightarrow \mathbf{DBimod}$  preserves adjunctions of the form  $F_* \dashv F^*$  for some dg functor  $F$ .

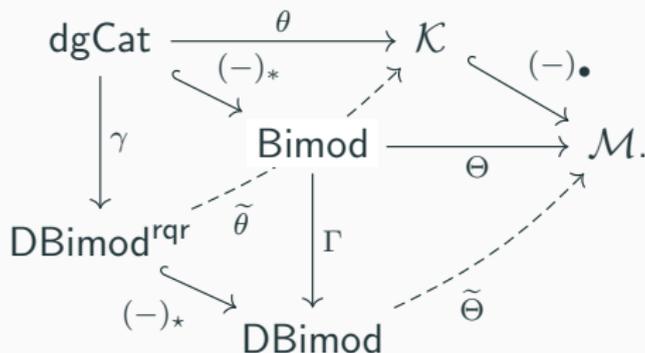
# A universal property

## Theorem (I.)

Consider another proarrow equipment  $(-)_\bullet: \mathcal{K} \leftrightarrow \mathcal{M}$ .  
Let  $\Theta: \text{Bimod} \rightarrow \mathcal{M}$  be a lax functor and  $\theta: \text{dgCat} \rightarrow \mathcal{K}$   
a pseudofunctor with  $\Theta \circ (-)_* = (-)_\bullet \circ \theta$  such that  $\Theta$  preserves  
adjunctions of the form  $F_* \dashv F^*$  for dg functors  $F$ . If

- $\Theta$  sends quasi-isomorphisms to invertible 2-cells and
- $\theta$  sends quasi-equivalences to equivalences,

then  $(\Theta, \theta)$  “factors through” the pair  $(\Gamma, \gamma)$ .



## A universal property

### Example

Consider the lax monoidal functor  $H^0: \text{Ch}(\mathbb{k}) \rightarrow \text{Mod}(\mathbb{k})$  taking the 0-th cohomology. This induces not only a 2-functor

$$H^0: \text{dgCat} = \text{Ch}(\mathbb{k})\text{-Cat} \rightarrow \text{Mod}(\mathbb{k})\text{-Cat},$$

but also a lax functor

$$\overline{H}^0: \text{Bimod} = \text{Ch}(\mathbb{k})\text{-Prof} \rightarrow \text{Mod}(\mathbb{k})\text{-Prof},$$

which sends a dg bimodule  $X: \mathcal{B}^{\text{op}} \otimes \mathcal{A} \rightarrow \text{Ch}_{\text{dg}}(\mathbb{k})$  to

$$H^0(\mathcal{B})^{\text{op}} \otimes H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B}^{\text{op}} \otimes \mathcal{A}) \xrightarrow{H^0(X)} H^0(\text{Ch}_{\text{dg}}(\mathbb{k})) \xrightarrow{H^0} \text{Mod}(\mathbb{k}).$$

Since  $\overline{H}^0$  sends quasi-isom to isom and  $H^0$  sends quasi-equiv to equiv,  $(\overline{H}^0, H^0)$  factors through  $(\Gamma, \gamma)$ .

### Remark

The pair  $(\Gamma, \gamma)$  forms an **equipment morphism** in the sense of [Verity 1992].

### Conjecture

The proarrow equipment  $(-)_* : \mathbf{DBimod}^{\text{rqr}} \hookrightarrow \mathbf{DBimod}$  is the localization of  $(-)_* : \mathbf{dgCat} \hookrightarrow \mathbf{Bimod}$  as a proarrow equipment in a suitable sense.

But there is no localization theory for proarrow equipments yet. (We may need to consider localizations of double categories.)

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## Adjunctions and equivalences\*

### Theorem (I.)

For a dg bimodule  $X: \mathcal{A} \rightarrow \mathcal{B}$ , TFAE:

- (1)  $X$  has a right adjoint in DBimod.
- (2)  $\mathbb{R}H_X: D(\mathcal{B}) \rightarrow D(\mathcal{A})$  preserves coproducts.
- (3)  $X$  is right compact; i.e.  $X(-, A) \in D(\mathcal{B})$  is compact for each  $A \in \mathcal{A}$ .

### Theorem (I.)

A dg bimodule  $X: \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence in DBimod

$\iff \mathbb{L}T_X: D(\mathcal{A}) \rightarrow D(\mathcal{B})$  is an equivalence.

### Theorem (I.)

A quasi-functor  $X: \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence in DBimod<sup>qr</sup>

$\iff$  both  $\mathbb{L}T_X: D(\mathcal{A}) \rightarrow D(\mathcal{B})$  and  $\mathbb{L}T_X|_{\overline{\mathcal{A}}}: \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$  are equivalences.