(Bi)Categorical Semantics for Non-Commutative Linear Logic

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Pacific Category Theory Seminar
March 27, 2026

Presentation Overview

Linear Logic

Multiplicative Linear logic

Girard introduced **linear logic**, a sub-structural (without contraction and weakening) logic in 1987 [12].

We shall be primarily concerned with the fragment of **multiplicative linear logic (MLL)**.

Connectives and Constants of MLL

	Multiplicative		Linear
Conjunction	$\otimes, \mathbf{1}$	Negation	$(-)^\perp$
Disjunction	$\parr, \perp$	Implication	$\multimap$

→ Multiplicative conjunction is otherwise known as **tensor** and disjunction as **par**.

Sequent Calculus for MLL

Logical Axioms

$$\vdash A, A^\perp$$

Structural Rules

$$\frac{\vdash A}{\vdash \sigma A}\ (\text{EXCH}) \qquad \frac{\vdash A, B \quad \vdash A^\perp, C}{\vdash B, C}\ (\text{CUT})$$

(where σA is obtained permuting the formulas of A)

Multiplicative Rules

$$\vdash \mathbf{1}\ (\text{axiom}) \qquad \frac{\vdash A}{\vdash \perp, A}\ \perp \qquad \frac{\vdash A, C \quad \vdash B, D}{\vdash A \otimes B, C, D}\ \otimes \qquad \frac{\vdash A, B, C}{\vdash A \parr B, C}\ \parr$$

Multiplicative Linear logic

Basic property of multiplicatives in MLL [12]:
Let A and B be formulas in the language of MLL.

- → Linear negation is involutive.

$$A^{\perp\perp} = A$$

- → The multiplicative constants are related by de Morgan duality.

$$\mathbf{1}^{\perp} = \perp \qquad \perp^{\perp} = \mathbf{1}$$

- → Any binary connective can be defined from any other and linear negation.

$$\begin{aligned} A \otimes B &= (A^{\perp} \parr B^{\perp})^{\perp} & A \parr B &= (A^{\perp} \otimes B^{\perp})^{\perp} \\ A \otimes B &= (A \multimap B^{\perp})^{\perp} & A \parr B &= A^{\perp} \multimap B \\ A \multimap B &= (A \otimes B^{\perp})^{\perp} & A \multimap B &= A^{\perp} \parr B \end{aligned}$$

- → Linear implication satisfies the following.

$$A^{\perp} = A \multimap \perp$$

- ⇒ Given tensor or par, linear implication may be defined using linear negation and, given additionally the constant $\perp$, vice versa.

Categorical Semantics for MLL

Categorical semantics for MLL were investigated by Seely in 1989:

Definition (Barr [2])

A $*$**-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, *)$ consists of:

- a symmetric monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$, with
- a full and faithful functor $(-)^* : \mathbb{X}^{op} \to \mathbb{X}$ such that there is a natural isomorphism

$$\mathbb{X}(A \otimes B, C^*) \cong \mathbb{X}(A, (B \otimes C)^*)$$

Proposition (Barr [2])

A $*$*-autonomous category* $(\mathbb{X}, \otimes, \mathbf{1}, *)$ *is closed, with internal hom defined by*

$$A \multimap B = (A \otimes B^*)^*$$

Proposition (Folklore)

Given a $*$*-autonomous category* $(\mathbb{X}, \otimes, \mathbf{1}, *)$, *define the following par structure by*

$$A \parr B = (A^* \otimes B^*)^* \qquad \bot = \mathbf{1}^*$$

then $(\mathbb{X}, \parr, \bot)$ *is a symmetric monoidal category.*

$*$-Autonomous Categories

There are many alternative definitions for $*$-autonomous categories, most importantly is the following:

Definition (Barr [3])

A **$*$-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \perp)$ consists of:

- a symmetric closed monoidal category $(\mathbb{X}, \otimes, \mathbf{1}, \multimap)$ with
- a **dualizing object** $\perp$, i.e. an object $\perp$ such that the canonical map

$$d_A : A \to (A \multimap \perp) \multimap \perp$$

is an isomorphism for all A.

Note: d_A is obtained by currying the evaluation map $ev_{A,\perp} : A \otimes (A \multimap \perp) \to \perp$.

Proof of equivalence of definitions (Sketch).

($\Rightarrow$) Let $A \multimap B = (A \otimes B^*)^*$ and $\perp = \mathbf{1}^*$.

($\Leftarrow$) Let $(-)^* = (-) \multimap \perp$. ☐

$\Rightarrow$ The first definition uses negation to define implication, while the the second uses implication and the constant $\perp$ to define negation.

$*$-Autonomous Categories

Example

1. Girard's category of coherent spaces and linear maps **Cohl** is $*$-autonomous, with dualizing object the singleton coherent space [19].
2. Category **Sup** of sup-lattices and sup-preserving maps, with dualizing object Ω^{op}.
3. Compact closed categories are $*$-autonomous with monoidal unit as the dualizing object.

Proposition (Dold, Puppe [11])

A $*$-autonomous category such that $(A \otimes B)^* \cong A^* \otimes B^*$ is compact closed.

4. Chu construction [6]

   Given a symmetric closed monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with pullbacks and some object $\perp \in \mathbb{X}$, let $\mathrm{Chu}(\mathbb{X}, \perp)$ denote the category of:
   - objects: triples (V, V', v), where V, V' are objects and $v : V \otimes V' \to \perp$ is a morphism in $\mathbb{X}$.
   - morphisms: $(f, g) : (V, V', v) \to (W, W', w)$ are pairs of morphisms $f : V \to W$ and $g : V' \to W'$ such that $(f \otimes 1_{W'}); w = (1_{V'} \otimes g); v$.

   Then, $\mathrm{Chu}(\mathbb{X}, \perp)$ is $*$-autonomous with $(\perp, \mathbf{1}, \rho_\perp)$ as dualizing object.

Cyclic Linear Logic

- In his original article, Girard provided two semantics for linear logic: coherent spaces and phase spaces.
- The phase space semantics already hinted that non-commutative linear logic was a possible direction for future research [12].

→ This lead Yetter to introduce a non-commutative variant of linear logic in 1990: **cyclic linear logic** [21].

→ In this talk, we will be interested in the multiplicative fragment, which we will now call **multiplicative cyclic linear logic (MCLL)**.

→ The sequent calculus for MCLL is the same as for MLL, except the *exchange rule* is replaced by the *cycling rule*:

$$\frac{\vdash A}{\vdash \sigma A}\ (\text{CYCL})$$

(where σA is obtained from A by a *cyclic* permutation)

→ The cycling rule is "forced" if we want a system with one notion of linear negation $(-)^{\perp}$.

Connectives and Constants of MCLL

Conjunction Disjunction	Multiplicative $\otimes, \mathbf{1}$ $\parr, \perp$	Negation Implication	Linear $(-)^{\perp}$ $\multimap, \multimapinv$

→ $\multimap$ will be known as **right** linear implication and $\multimapinv$ as **left** linear implication.

Cyclic Linear logic

Basic property of multiplicatives in MCLL:
Let A and B be formulas in the language of MCLL.

- Linear negation is involutive.

$$A^{\perp\perp} = A$$

- The multiplicative constants are related by de Morgan duality.

$$\mathbf{1}^{\perp} = \perp \qquad \perp^{\perp} = \mathbf{1}$$

- Any binary connective can be defined from any other and linear negation.

$$\begin{aligned}
A \otimes B &= (B^{\perp} \parr A^{\perp})^{\perp} = (B \multimap A^{\perp})^{\perp} = (B^{\perp} \multimapinv A)^{\perp}\\
A \parr B &= (B^{\perp} \otimes A^{\perp})^{\perp} = A^{\perp} \multimap B = A \multimapinv B^{\perp}\\
A \multimap B &= (B^{\perp} \otimes A)^{\perp} = A^{\perp} \parr B = A^{\perp} \multimapinv B^{\perp}\\
B \multimapinv A &= (A \otimes B^{\perp})^{\perp} = B \parr A^{\perp} = B^{\perp} \multimap A^{\perp}
\end{aligned}$$

- Right and left linear implication satisfies the following.

$$A^{\perp} = A \multimap \perp \qquad A^{\perp} = \perp \multimapinv A$$

Bilinear Logic

- If the cycling rule is dropped, there are necessarily two notions of linear negation.
- This variant of non-commutative MLL without the exchange rule has been studied by different authors and given various names.

→ In 1958, Lambek introduced a sub-structural deductive system known at the time as the **syntactic calculus**, in view of applications in linguistics [17].

→ In 1993, Lambek unified this syntactic calculus with linear logic and developed **bilinear logic** [15, 16], which in fact borrows from earlier work by Grishin in 1983 [13].

→ Around the same time, in 1991, Abrusci developed **noncommutative linear propositional logic** to investigate linear logic without the exchange rule, with a focus on non-commutative phase spaces [1].

→ In this talk, we will be interested in the multiplicative fragment, which we will now call **multiplicative bilinear logic (MBLL)**, which essentially amounts to MLL without the exchange rule, leading to two notions of linear negation and implication.

Connectives and Constants of MBLL

	Multiplicative		Linear
Conjunction	$\otimes, \mathbf{1}$	Negation	$(-)^{\perp}, {}^{\perp}(-)$
Disjunction	$\parr, \perp$	Implication	$\multimap, \multimapinv$

→ $(-)^{\perp}$ will be known as **right** linear negation and ${}^{\perp}(-)$ as **left** linear negation.

Bilinear Linear logic

Basic property of multiplicatives in MBLL [1]:
Let A and B be formulas in the language of MBLL.

- Linear negation is involutive.

$$({}^{\perp}A)^{\perp} = A = {}^{\perp}(A^{\perp})$$

- The multiplicative constants are related by de Morgan duality.

$$\mathbf{1}^{\perp} = \perp = {}^{\perp}\mathbf{1} \qquad \perp^{\perp} = \mathbf{1} = {}^{\perp}\perp$$

- Any binary connective can be defined from any other with right and left linear negation.

$$\begin{aligned}
A \otimes B &= {}^{\perp}(B^{\perp} \parr A^{\perp}) = ({}^{\perp}B \parr {}^{\perp}A)^{\perp} = {}^{\perp}(B \multimap A^{\perp}) = ({}^{\perp}B \multimapinv A)^{\perp} \\
A \parr B &= {}^{\perp}(B^{\perp} \otimes A^{\perp}) = ({}^{\perp}B \otimes {}^{\perp}A)^{\perp} = {}^{\perp}A \multimap B = A \multimapinv B^{\perp} \\
A \multimap B &= ({}^{\perp}B \otimes A)^{\perp} = A^{\perp} \parr B = A^{\perp} \multimapinv B^{\perp} \\
B \multimapinv A &= {}^{\perp}(A \otimes B^{\perp}) = B \parr {}^{\perp}A = {}^{\perp}B \multimap {}^{\perp}A
\end{aligned}$$

- Right and left linear implication satisfy the following.

$$A^{\perp} = A \multimap \perp \qquad {}^{\perp}A = \perp \multimapinv A$$

- The **Grishin laws** holds [15].

$$(A \multimap B) \parr C = A \multimap (B \parr C) \qquad A \parr (B \multimapinv C) = (A \parr B) \multimapinv C$$

Categorical Semantics

Non-Symmetric $*$-Autonomous Categories

In 1995, Barr extended $*$-autonomous categories to capture **multiplicative bilinear logic**.

Definition (Barr [4])

A **non-symmetric $*$-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, (-)^*, {}^*(-))$ consists of:

- a monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with
- an equivalence $(-)^* : \mathbb{X} \to \mathbb{X}^{op}$ such that there is a natural isomorphism
$$\mathbb{X}(A, B^*) \cong \mathbb{X}(\mathbf{1}, (A \otimes B)^*)$$

→ As an equivalence $(-)^* : \mathbb{X} \to \mathbb{X}^{op}$, there is a functor ${}^*(-) : \mathbb{X}^{op} \to \mathbb{X}$ such that

$${}^*(A^*) \cong A \cong ({}^*A)^*$$

which is the second form of linear negation.

- In fact, the second bullet can be reformulated to make express mention of the two negations:

an equivalence $\mathbb{X} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathbb{X}^{op}$ such that there is a natural isomorphism

$$\mathbb{X}(A \otimes B, C) \cong \mathbb{X}(A, (B \otimes {}^*C)^*) \quad \text{or} \quad \mathbb{X}(A \otimes B, C) \cong \mathbb{X}(B, {}^*(C^* \otimes A))$$

Warning: Given the above formulation, $(-)^*$ models left linear negation ${}^\perp(-)$ and ${}^*(-)$ models right linear negation $(-)^\perp$.

Non-Symmetric $*$-Autonomous Categories

Proposition (Barr [2])

A non-symmetric $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ *is biclosed, with internal homs defined by*

$$A \multimap B = {}^*(B^* \otimes A) \qquad B \circ\!- A = (A \otimes {}^*B)^*$$

As you would expect, the definition can also be given in terms of implication.

Definition (Barr [4])

A **non-symmetric $*$-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \circ\!-, \bot)$ consists of:

- a biclosed monoidal category $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \circ\!-)$ with
- a **dualizing object** $\bot$, i.e. an object $\bot$ such that the canonical maps

  $$d^1_A : A \to \bot \circ\!- (A \multimap \bot) \qquad d^2_A : A \to (\bot \circ\!- A) \multimap \bot$$

  are isomorphisms for all A.

Note: d^1_A is obtained by left currying the evaluation map $ev^R_{A,\bot} : A \otimes (A \multimap \bot) \to \bot$ and d^2_A by right currying the coevaluation map $ev^L_{\bot,A} : (\bot \circ\!- A) \otimes A \to \bot$.

From $*$-Autonomous to Linearly Distributive

Proposition (Folklore)

Given a non-symmetric $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$, define the following

$$A \parr B = {}^*(B^* \otimes A^*) \qquad \bot = {}^*\mathbf{1} \quad \textit{or equivalently} \quad A \parr B = ({}^*B \otimes {}^*A)^* \qquad \bot = \mathbf{1}^*$$

then $(\mathbb{X}, \parr, \bot)$ is a monoidal category.

Can we consider alternative semantics which put conjunction and disjunction on a “level playing field”?

Cockett and Seely introduced alternative semantics for MLL (with and without negation) in 1992: **linearly distributive categories** [9].

Motivation:

- → take multiplicative conjunction $\otimes$ and disjunction $\parr$ as primitive
- → provide a model for a minimal logic with two connectives, two constants, introductions rules and Gentzen’s cut rule (non-commutative MLL without linear negation and linear implication)
- → promote the importance of the distributivity between tensor and par

Linearly Distributive Categories

Definition (Cockett, Seely [9])

A **linearly distributive category**, or **LDC**, $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a **tensor** monoidal structure $(\mathbb{X}, \otimes, \mathbf{1})$,
- a **par** monoidal structure $(\mathbb{X}, \parr, \bot)$, and
- left and right **linear distributivity** natural transformations

$$\delta^R_{A,B,C} \colon (A \parr B) \otimes C \to A \parr (B \otimes C)$$

$$\delta^L_{A,B,C} \colon A \otimes (B \parr C) \to (A \otimes B) \parr C$$

satisfying coherence conditions.

⇒ LDC provide the categorical semantics for non-commutative MLL without linear negation and linear implication.

Remark: Notational conflict

	Tensor	Par
Cockett, Seely	$\otimes, \top$	$\oplus, \bot$
Girard	$\otimes, \mathbf{1}$	$\parr, \bot$

Linearly Distributive Categories

Example

1. Every monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ can be viewed as a LDC, when taking

$$\otimes = \parr \qquad \mathbf{1} = \bot \qquad \delta^R_{A,B,C} = \alpha_{\otimes A,B,C} \qquad \delta^L_{A,B,C} = \alpha_{\otimes A,B,C}^{-1}$$

   These LDCs $(\mathbb{X}, \otimes, \mathbf{1}, \otimes, \mathbf{1})$ are known as **degenerate**.

2. Every (standard or non-symmetric) $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ is a LDC. It is now a standard exercise in linear logic to determine the linear distributivities in this case.

3. Consider a monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with an object $\bot$, which has a $\otimes$-inverse $\bot^{-1}$, i.e. maps

$$s^L : \bot \otimes \bot^{-1} \to \mathbf{1} \qquad s^R : \bot^{-1} \otimes \bot \to \mathbf{1}$$

   satisfying a coherence condition and define the **shifted tensor**

$$A \parr B = A \otimes (\bot^{-1} \otimes B)$$

   Then, $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ is a LDC with invertible linear distributivities [9].

Adding Negation

In order to model **multiplicative bilinear logic** and recover non-symmetric $*$-autonomous categories, linear negation must be considered in the context of LDCs.

Definition (Cockett, Seely [10])

Given a LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$, a **complementation pair** (A, A^c, γ, τ) consists of:

- an object A, called the left complement,
- an object A^c, called the right complement, and
- a pair of evaluation and coevaluation maps

$$\gamma : A \otimes A^c \to \bot \qquad \tau : \mathbf{1} \to A^c \parr A$$

such that the following diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "u^R_{\otimes A}"'] & A \\
A \otimes \mathbf{1} \arrow[d, "1_A \otimes \tau"'] & \bot \otimes A \arrow[u, "u^L_{\parr A}"'] \\
A \otimes (A^c \parr A) \arrow[r, "\delta^L_{A,A^c,A}"'] & (A \otimes A^c) \parr A \arrow[u, "\gamma \parr 1_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A^c \arrow[r, "1_{A^c}"] \arrow[d, "u^L_{\otimes A^c}"'] & A^c \\
\mathbf{1} \otimes A^c \arrow[d, "\tau \otimes 1_{A^c}"'] & A^c \parr \bot \arrow[u, "u^R_{\parr A^c}"'] \\
(A^c \parr A) \otimes A^c \arrow[r, "\delta^R_{A^c,A,A^c}"'] & A^c \parr (A \otimes A^c) \arrow[u, "1_{A^c} \parr \gamma"']
\end{tikzcd}$$

Adding Negation

Lemma (Cockett, Seely [9])

Let (A, A^c, γ, τ) *be a complementation pair in a LDC* $\mathbb{X}$*, then there are adjunctions*

$$A \otimes (-) \dashv A^c \parr (-) \qquad (-) \otimes A^c \dashv (-) \parr A$$

- → We may see A^c as essentially the right linear negation of A, and dually that A is the left linear negation of A^c.
- → So, a LDC has negation if there is a coherent way of associating to every object both a left and right linear negation.

Definition (Cockett, Seely [9])

A **LDC with negation** $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot, (-)^\bot, {}^\bot(-))$ is a LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ with object functions

$$(-)^\bot, {}^\bot(-) : \mathbb{X} \to \mathbb{X}$$

together with the following parametrized family of maps

$$\begin{aligned} \gamma_A^R &: A \otimes A^\bot \to \bot & \qquad \tau_A^L &: \mathbf{1} \to A^\bot \parr A \\ \gamma_A^L &: {}^\bot A \otimes A \to \bot & \qquad \tau_A^R &: \mathbf{1} \to A \parr {}^\bot A \end{aligned}$$

such $(A, A^\bot, \gamma_A^R, \tau_A^L)$ and $({}^\bot A, A, \gamma_A^L, \tau_A^R)$ form complementation pairs.

Adding Implication

What about defining linear negation in terms of linear implication and $\bot$?

Definition (Cockett, Seely [8])

A **Grishin LDC** $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot, \multimap, \multimapinv)$ consists of:

- a LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$, whose
- $\otimes$-monoidal structure is closed $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \multimapinv)$, such that
- the following canonical maps are isomorphisms for all A, B and C in $\mathbb{X}$.

$$(A \multimap B) \parr C \to A \multimap (B \parr C) \qquad A \parr (B \multimapinv C) \to (A \parr B) \multimapinv C$$

Note: The above canonical maps are obtained by currying the following composites:

$$A \otimes ((A \multimap B) \parr C) \xrightarrow{\delta^L_{A,A \multimap B,C}} (A \otimes (A \multimap B)) \parr C \xrightarrow{ev_{A,B} \parr 1_C} B \parr C$$

$$(A \parr (B \multimapinv C)) \otimes C \xrightarrow{\delta^R_{A,B \multimapinv C,C}} A \parr ((B \multimapinv C) \otimes C) \xrightarrow{1_A \parr ve_{B,C}} A \parr B$$

Categorical Semantics for MBLL

→ Altogether, the categorical semantics for multiplicative bilinear logic are given by:

Proposition (Cockett, Seely [8])

The following notions coincide:

1. *non-symmetric $*$-autonomous categories* $(\mathbb{X}, \otimes, \mathbf{1}, (-)^*, {}^*(-))$,
2. *non-symmetric $*$-autonomous categories* $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \multimapinv, \bot)$,
3. *linearly distributive categories with negation* $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot, (-)^\bot, {}^\bot(-))$*, and*
4. *Grishin linearly distributive categories* $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot, \multimap, \multimapinv)$.

Proof (Sketch).

$(1 \Leftrightarrow 2)$

$$A \multimap B = {}^*(B^* \otimes A) \qquad B \multimapinv A = (A \otimes {}^*B)^* \qquad \bot = \mathbf{1}^* \cong {}^*\mathbf{1}$$

$$A^* = \bot \multimapinv A \qquad {}^*A = A \multimap \bot$$

$(1 \Leftrightarrow 3)$

$$A \parr B = {}^*(B^* \otimes A^*) \cong ({}^*B \otimes {}^*A)^* \qquad \bot = {}^*\mathbf{1} \cong \mathbf{1}^*$$

$$(-)^* = {}^\bot(-) \qquad {}^*(-) = (-)^\bot$$

$(3 \Leftrightarrow 4)$

$$A \multimap B = ({}^\bot B \otimes A)^\bot \cong A^\bot \parr B \qquad B \multimapinv A = {}^\bot(A \otimes B^\bot) = B \parr {}^\bot A$$

$$A^\bot = A \multimap \bot \qquad {}^\bot A = \bot \multimapinv A$$

Categorical Semantics for MBLL

Example

1. The posetal category of relations on a set X, Rel(X), is a non-symmetric $*$-autonomous category [4]:
   - monoidal product is standard relational composition: given relations $R, S : X \to X$, define
   $$R \otimes S = \{(x, x'') \mid \exists x' \in X, (x, x') \in R \wedge (x', x'') \in S\}$$
   - dualizing object is inequality relation: $\Delta_X^* = \{(x, x') \mid x \neq x'\}$
   - left and right linear negation is complement inverse: given relation $R : X \to X$, define $R^\perp = {}^\perp R : X \to X$ by
   $$R^\perp = \{(x, x') \mid (x', x) \notin R\}$$
2. A non-commutative classical phase space, in the sense of Abrusci [1], is a non-symmetric $*$-autonomous category.
3. The Chu construction was extended to non-symmetric biclosed monoidal categories by Barr [4] to produce non-symmetric $*$-autonomous categories.

→ Many important examples of categorical semantics for MBLL in fact have one notion of linear negation ⇒ multiplicative cyclic linear logic (MCLL).

Cyclic $*$-Autonomous Categories

In 1990, Yetter introduced **cyclic linear logic** and gave quantale semantics for it to capture non-commutative phase spaces.

General categorical semantics for were given by Rosenthal in 1994.

Definition (Rosenthal [18])

Consider a biclosed monoidal category $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \multimapinv)$.

- An object $\bot$ is **dualizing** if the following canonical maps are isomorphims for all A.

$$d^1_A : A \to \bot \multimapinv (A \multimap \bot) \qquad d^2_A : A \to (\bot \multimapinv A) \multimap \bot$$

- An object $\bot$ is **cyclic** if there is an isomorphism

$$\theta_A : \bot \multimapinv A \cong A \multimap \bot$$

  such that the following diagram commutes.

$$\begin{tikzcd}
 & \bot \multimapinv (A \multimap \bot) \arrow[d, "\bot \multimapinv \theta_A"] \\
A \arrow[ur, "d^1_A"] \arrow[dr, "d^2_A"'] & \bot \multimapinv (\bot \multimapinv A) \arrow[d, "\theta_{\bot \multimapinv A}"] \\
 & (\bot \multimapinv A) \multimap \bot
\end{tikzcd}$$

Categorical Semantics for MCLL

→ The categorical semantics for cyclic multiplicative linear logic are given by:

Definition (Rosenthal [18])

A **cyclic $*$-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \multimapinv, \bot)$ consists of:

- a biclosed monoidal category $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \multimapinv)$, with
- a cyclic dualizing object $\bot$.

Note: The definition of a cyclic $*$-autonomous category in terms of linear negation as well as the linearly distributive variants do not appear in the literature, although it should be straightforward in view of the bicategorical semantic developments.

Example

1. The posetal category of relations on a set X, $\mathsf{Rel}(X)$, is a cyclic $*$-autonomous category [4].
2. A non-commutative phase space, in the sense of Yetter [21], is a cyclic $*$-autonomous category.
3. The Chu construction was extended to a non-symmetric biclosed monoidal category by Barr, further considering the distinguished object to be cyclic [4].

Bicategorical Semantics

Bicategorical Semantics

To model non-commutative logical connectives:

- → We can use non-symmetric monoidal products.
- → However, the most “natural” perspective is to consider bicategorical composition.
- → It is intrinsically a non-commutative operation.

Example

The non-symmetric $*$-autonomous category Rel(X) models multiplicative conjunction by composition.

⇒ It is with this mindset that Cockett, Kowslowski and Seely introduced **linear bicategories** and **cyclic $*$-autonomous bicategories** in 2000 and 2001, to provide bicategorical semantics for MLL, MBLL and MCLL.

Linear Bicategories

Definition (Cockett, Kowslowski and Seely [7])

A **linear bicategory** $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X)$ consists of:

- a class of 0-cells $\mathcal{B}_0$
- a category of 1-cells and 2-cells $\mathcal{B}_1$, with functors $\mathrm{dom}, \mathrm{cod} : \mathcal{B}_1 \to \mathcal{B}_0$, where the following full subcategories are denoted by

$$\mathcal{B}(X, Y) = \{f \in \mathcal{B}_1 \mid \mathrm{dom}(f) = X, \mathrm{cod}(f) = Y\}$$

- a **tensor** bicategorical structure $(\mathcal{B}, \otimes, \mathbf{1}_X)$ with composition functor and identity 1-cells

$$\otimes : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \qquad \mathbf{1}_X : X \to X$$

- a **par** bicategorical structure $(\mathcal{B}, ⅋, \bot_X)$ with composition functor and identity 1-cells

$$⅋ : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \to \mathcal{B}(X, Z) \qquad \bot_X : X \to X$$

- left and right **linear distributivity** transformations

$$\delta^R_{R,S,T} : (R ⅋ S) \otimes T \to R ⅋ (S \otimes T) \qquad \delta^L_{R,S,T} : R \otimes (S ⅋ T) \to (R \otimes S) ⅋ T$$

satisfying coherence conditions.

Linear Bicategories

Example

1. Every bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X)$ can be viewed as a linear bicategory, when taking

$$\otimes = \parr \qquad \mathbf{1} = \bot \qquad \delta^R_{A,B,C} = \alpha_{\otimes A,B,C} \qquad \delta^L_{A,B,C} = \alpha_{\otimes}{}^{-1}_{A,B,C}$$

2. Given a LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$, its suspension $(\mathcal{B}(\mathbb{X}), \otimes, \mathbf{1}_*, \parr, \bot_*)$ is a linear bicategory.
3. The locally posetal bicategory of sets and relations, **Rel**, is a linear bicategory: given relations $R : X \to Y$ and $S : Y \to Z$,

$$R \otimes S = \{(x, z) \mid \exists y \in Y, (x, y) \in R \wedge (y, z) \in S\}$$

$$R \parr S = \{(x, z) \mid \forall y \in Y, (x, y) \in R \vee (y, z) \in S\}$$

$$\mathbf{1}_X = \Delta_X = \{(x, x') \mid x = x'\} \qquad \bot_X = \Delta^*_X = \{(x, x') \mid x \neq x'\}$$

Then, it is fairly straightforward to check that for $R : X \to Y$, $S : Y \to Z$ and $T : Z \to W$,

$$R \otimes (S \parr T) \subseteq (R \otimes S) \parr T \qquad (R \parr S) \otimes T \subseteq R \parr (S \otimes T)$$

Linear Bicategories

Example

④ Consider a LD-quantale $(Q, \otimes, \mathbf{1}, \parr, \bot)$, then define Q-**Rel** by

- 0-cells: sets X, Y
- 1-cells: Q-valued relation $X \to Y$ are functions $R : X \times Y \to Q$
- 2-cells: posetal inherited from Q

Then, given relations $R : X \to Y$ and $S : Y \to Z$, define

$$(R \otimes S)(x, z) = \bigvee_{y \in Y} R(x, y) \otimes S(y, z) \qquad (R \parr S)(x, z) = \bigwedge_{y \in Y} R(x, y) \parr S(y, z)$$

This defines a locally posetal linear bicategory [5].

⑤ Given a LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ with linear products and coproducts, in the sense of [10], define the bicategory **Matrix**$(\mathbb{X})$ by

- 0-cells: natural numbers
- 1-cells: $A : m \to n$ is an $m \times n$ matrix $[A_{ij}]$ of objects in $\mathbb{X}$
- 2-cells: $f : A \Rightarrow B$ is a $m \times n$ matrix $[f_{ij}]$ of maps $f_{ij} : A_{ij} \to B_{ij}$ in $\mathbb{X}$

Then, given $A : m \to n$ and $B : n \to p$, define

$$[A_{ij}] \otimes [B_{jk}] = [\sum (A_{ij} \otimes B_{jk})] \qquad [A_{ij}] \parr [B_{jk}] = [\prod (A_{ij} \parr B_{jk})]$$

This defines a linear bicategory [7].

Adding Negation

As with bicategories, the concept of “adjoint” 1-cells in linear bicategories is crucial.

Definition (Cockett, Koslowski, Seely [7])

A **linear adjunction**

$$v = \langle \tau, \gamma \rangle \colon R \dashv\!\!\mid S \colon X \to Y$$

consists of:

- a 1-cell $R \colon X \to Y$, known as the **left linear adjoint**,
- a 1-cell $S \colon Y \to X$, known as the **right linear adjoint**, and
- a pair of **unit** and **counit** 2-cells $\tau : \mathbf{1}_X \Rightarrow R \parr S$ and $\gamma : S \otimes R \Rightarrow \bot_Y$

such that the following snake equations commute.

$$\begin{tikzcd}
S \arrow[rr, "1_S"] \arrow[d, "u^R_{\otimes\, S}"'] & & S \\
S \otimes \mathbf{1}_X \arrow[d, "1_S \otimes \tau"'] & & \bot_Y \otimes S \arrow[u, "u^L_{\parr\, S}"'] \\
S \otimes (R \parr S) \arrow[r, "\delta^L_{S,R,S}"'] & (S \otimes R) \parr S \arrow[r, phantom, ""] & \arrow[u, "\gamma \parr 1_S"']
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[rr, "1_R"] \arrow[d, "u^L_{\otimes\, S}"'] & & R \\
\mathbf{1}_X \otimes R \arrow[d, "\tau \otimes 1_R"'] & & R \parr \bot_Y \arrow[u, "u^R_{\parr\, R}"'] \\
(R \parr S) \otimes R \arrow[rr, "\delta^R_{R,S,R}"'] & & R \parr (S \otimes R) \arrow[u, "1_R \parr \gamma"']
\end{tikzcd}$$

Adding Negation

→ Linear adjoints are the bicategorical analogue of the complementation pair.

Lemma

Given a LDC $(\mathbb{X}, \otimes, \mathbf{1}, ⅋, \bot)$ *and consider a complementation pair* (A, A^c, γ, τ)*, then this is a linear adjunction*

$$v = \langle \tau, \gamma \rangle : A^c \dashv\!\!\vert A : * \to *$$

in its suspension as linear bicategory $(\mathcal{B}(\mathbb{X}), \otimes, \mathbf{1}_*, ⅋, \bot_*)$.

→ Left linear adjoints model right negation, while right linear adjoints left negation in MBLL.

Definition

A **linear bicategory with negation** $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X, (-)^\bot, {}^\bot(-))$ consists of:

- a linear bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X)$ such that
- every 1-cell $S : X \to Y$ has a chosen left and right linear adjoints

$$v^R = \langle \tau_S^R, \gamma_S^R \rangle : S^\bot \dashv\!\!\vert S : Y \to X \qquad v^L = \langle \tau_S^L, \gamma_S^L \rangle : S \dashv\!\!\vert {}^\bot S : X \to Y$$

Lemma

Given a LDC with negation $(\mathbb{X}, \otimes, \mathbf{1}, ⅋, \bot, (-)^\bot, {}^\bot(-))$*, then its suspension is a linear bicategory with negation* $(\mathcal{B}(\mathbb{X}), \otimes, \mathbf{1}_*, ⅋, \bot_*, (-)^\bot, {}^\bot(-))$.

Adding Implication

What about defining negation in terms of implication in the context of linear bicategories? We need to consider **closure in bicategories**.

Definition (Street [20])

Let $(\mathcal{B}, \otimes, \mathbf{1}_X)$ be a bicategory.

- A **right hom** from $R : X \to Y$ to $S : X \to Z$ (or right extension of S along R) in $\mathcal{B}$ is a 1-cell, equipped with a 2-cell

  $$R \multimap S : Y \to Z \qquad ev_{R,S} : R \otimes (R \multimap S) \Rightarrow S$$

  denoted together by $\langle R \multimap S, ev_{R,S} \rangle$, satisfying the following universal property:

  for any 1-cell $T\colon Y \to Z$ and any 2-cell $f\colon R \otimes T \Rightarrow S$, there exists a unique 2-cell $curry(f) : T \to R \multimap S$ such that $(1_R \otimes curry(f)); ev_{R,S} = f$.

  Or equivalently, it is a 1-cell $R \multimap S : Y \to Z$ such that there is a natural isomorphism

  $$\mathcal{B}(X, Z)(R \otimes -, S) \cong \mathcal{B}(Y, Z)(-, R \multimap S)$$

- A **left hom** in $\mathcal{B}$ is a right hom in $\mathcal{B}^{op}$.

Adding Implication

Definition (Cockett, Koslowski, Seely [7])

Let $(\mathcal{B}, \otimes, \mathbf{1}_X, \parr, \bot_X)$ be a linear bicategory. A right hom $\langle R \multimap S, ev_{R,S} \rangle$ from $R : X \to Y$ to $S : X \to Z$ in $\mathcal{B}$ is **linear** if it is **absolute** with respect to $\parr$:

if $\langle (R \multimap S) \parr T, ev_{R,S} \parr 1_T \rangle$ is a right hom from R to $S \parr T$ for any 1-cell $T : Z \to W$.

Dually for left homs.

A **closed linear bicategory** $(\mathcal{B}, \otimes, \mathbf{1}_X, \parr, \bot_X, \multimap, \multimapinv)$ consists of:

- a linear bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X, \parr, \bot_X)$ such that
- given 1-cells $R\colon X \to Y$, $S\colon X \to Z$ and $T\colon W \to Y$, there is chosen right and left linear homs

$$\langle R \multimap S, ev_{R,S} \rangle \qquad \langle T \multimapinv R, ve_{T,R} \rangle$$

$$\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "S"'] \arrow[dr, phantom, "\overset{ev_{R,S}}{\Leftarrow}"] & Y \arrow[dl, "R \multimap S"] \\
Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
& W \arrow[dl, "T \multimapinv R"'] \arrow[d, "T"] \\
X \arrow[r, "R"'] \arrow[ur, phantom, "\overset{ve_{T,R}}{\Rightarrow}"] & Y
\end{tikzcd}$$

Adding Implication

Lemma (Cockett, Koslowski, Seely [7])

Given 1-cells $R : X \to Y$, $S : X \to Z$, $T : Z \to W$, *and right hom* $\langle R \multimap S, ev_{R,S} \rangle$ *in a linear bicategory, then the following are equivalent:*

1. $\langle (R \multimap S) \parr T, ev_{R,S} \parr 1_T \rangle$ *is a right hom.*
2. *The following canonical 2-cell is invertible.*

$$(R \multimap S) \parr T \Rightarrow R \multimap (S \parr T)$$

In other words, a right hom $\langle R \multimap S, ev_{R,S} \rangle$ *is linear if and only if the above canonical map is invertible for every 1-cell* $T : Z \to W$. *Dually, for left homs.*

Remark: This is the bicategorical analogue of the Grishin laws.

Lemma

Given a Grishin LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot, \multimap, \multimapinv)$*, then its suspension is a closed linear bicategory* $(\mathcal{B}(\mathbb{X}), \otimes, \mathbf{1}_*, \parr, \bot_*, \multimap, \multimapinv)$.

Bicategorical Semantics for MBLL

→ Altogether, the bicategorical semantics for multiplicative bilinear logic are given by:

Proposition (Cockett, Koslowski, Seely [7])

The following are equivalent:

1. *linear bicategories with negation* $(\mathcal{B}, \otimes, \mathbf{1}_X, \parr, \perp_X, (-)^\perp, {}^\perp(-))$*, and*
2. *closed linear bicategories* $(\mathcal{B}, \otimes, \mathbf{1}_X, \parr, \perp_X, \multimap, \multimapinv)$.

Proof Sketch.

(1 ⇒ 2) Consider 1-cell $R : X \to Y$, then

$$R \dashv\!\!\dashv \perp_Y \multimapinv R = {}^\perp R : X \to Y \qquad R^\perp = R \multimap \perp_X \dashv\!\!\dashv R : Y \to X$$

(2 ⇒ 1) Let $R : X \to Y$, $S : X \to Z$ and $T : W \to Y$ be 1-cells, then

$$R \multimap S = R^\perp \parr S : Y \to Z \qquad T \multimapinv R = T \parr {}^\perp R : W \to X$$

where $R \dashv\!\!\dashv {}^\perp R$ and $R^\perp \dashv\!\!\dashv R$. ☐

Note: The definition of a $*$-autonomous bicategory in terms of linear negation and linear implication does not appear explicitly in the literature, although it is alluded to in [7].

Bicategorical Semantics for MBLL

Example

1. The linear bicategory **Rel** has negation: given relation $R : X \to Y$, define $R^\perp = {}^\perp R : Y \to X$ by taking the complement and opposite
$$R^\perp = \{(y, x) \mid (x, y) \notin R\}$$
2. The linear bicategory Q-**Rel** has negation if Q is a Girard quantale (e.g. a Boolean algebra) [5]: given Q-relation $R : X \times Y \to Q$, define $R^\perp = {}^\perp R : Y \times X \to Q$ pointwise by
$$R^\perp(y, x) = (R(x, y))^\perp$$
3. The linear bicategory **Matrix**$(\mathbb{X})$ has negation if $\mathbb{X}$ is a $*$-autonomous category [10]: given $A : m \to n$, i.e. $m \times n$ matrix $[A_{ij}]$, define $A^\perp = {}^\perp A : n \to m$ as $n \times m$ matrix defined by
$$A^\perp_{ij} = (A_{ji})^\perp$$

→ Examples 2 and 3 can be easily extended to have distinct left and right linear negations if the base structures Q and $\mathbb{X}$ do.

→ However, most "natural" examples of $*$-autonomous categories are symmetric or cyclic.

⇒ Let us look at bicategorical semantics for MCLL.

Cyclic $*$-Autonomous Bicategories

In order to consider a bicategorical cyclic Chu construction, Koslowski introduced the bicategorical analogue of the cyclic $*$-autonomous category in 2001, around the same time linear bicategories were introduced.

Definition (Koslowski [14])

A **cyclic $*$-autonomous bicategory** $(\mathcal{B}, \otimes, \mathbf{1}_X, *)$ consists of a bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X)$ such that

1. for any pair of 0-cells X, Y, there is an adjoint equivalence
$$(-)^* \dashv ((-)^*)^{op} : \mathcal{B}(X, Y) \to \mathcal{B}(Y, X)^{op}$$
2. for any 1-cell $R : X \to Y$, the 1-cell R^* is the right hom from R to $\mathbf{1}_X^*$, i.e.
$$R^* = R \multimap \mathbf{1}_X^* : Y \to X$$
such that these homs are natural in R.

The second condition can be alternatively be stated as follows:

- for any triple of 0-cells X, Y and Z, there are natural isomorphisms
$$\mathcal{B}(X, Z)(R \otimes S, T^*) \cong \mathcal{B}(Y, Z)(S, (T \otimes R)^*)$$

Cyclic $*$-Autonomous Bicategories

As expected, the definition can be reformulated in terms of cyclic dualizing 1-cells.

Definition (Cockett, Koslowski, Seely [7], Blute, Kudzman-Blais, Niefield [5])

Suppose $\mathcal{D} = \{\bot_X \colon X \to X \mid X \in \mathcal{B}\}$ is a family of 1-cells in a closed bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X, \multimap, \multimapinv)$. Given $R \colon X \to Y$, consider the following canonical 2-cells

$$\delta_{R,Y} : R \Rightarrow (\bot_Y \multimapinv R) \multimap \bot_Y \qquad \delta_{X,R} : R \Rightarrow \bot_X \multimapinv (R \multimap \bot_X)$$

- A family $\mathcal{D}$ is called **dualizing** if the 2-cells $\delta_{R,Y}$ and $\delta_{X,R}$ are invertible, for all $R : X \to Y$.
- A dualizing family $\mathcal{D}$ is called **cyclic** if there are invertible 2-cells, atural in $R : X \to Y$

  $$\theta_R \colon \bot_Y \multimapinv R \xrightarrow{\sim} R \multimap \bot_X \quad \text{such that}$$

  natural in $R : X \to Y$

  $$\begin{tikzcd}
  & \bot_X \multimapinv (R \multimap \bot_X) \arrow[d, Rightarrow, "\bot_X \multimapinv \theta_R"] \\
  R \arrow[ur, Rightarrow, "\delta_{X,R}"] \arrow[dr, Rightarrow, "\delta_{R,Y}"'] & \bot_X \multimapinv (\bot_Y \multimapinv R) \arrow[d, Rightarrow, "\theta_{\bot_Y \multimapinv R}"] \\
  & (\bot_Y \multimapinv R) \multimap \bot_Y
  \end{tikzcd}$$

- A cyclic $*$-autonomous bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X, \multimap, \multimapinv, \bot_X)$ is a closed bicategory $(\mathcal{B}, \otimes, \mathbf{1}_X, \multimap, \multimapinv)$ with a cyclic dualizing family $\mathcal{D} = \{\bot_X\}$.

Bicategorical Semantics for MCLL

Cockett, Koslowski and Seely also develop the notion of **$*$-linear bicategories**, which are linear bicategories with one coherent notion of negation, which are:

- closed linear bicategories $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X, \multimap, \multimapinv)$ with a natural isomorphism

$$\alpha_R : \bot_Y \multimapinv R \Rightarrow R \multimap \bot_X$$

  satisfying coherence conditions, or equivalently
- linear bicategories with coherent **cyclic linear adjoints** $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X, (-)^\bot)$.

→ Altogether, the bicategorical semantics for multiplicative cyclic linear logic are given by:

Proposition (Cockett, Koslowski, Seely [7])

The following notions coincide:

1. *cyclic $*$-autonomous bicategories* $(\mathcal{B}, \otimes, \mathbf{1}_X, *)$
2. *cyclic $*$-autonomous bicategories* $(\mathcal{B}, \otimes, \mathbf{1}_X, \multimap, \multimapinv, \bot_X)$,
3. *$*$-linear bicategories* $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X, (-)^\bot)$, *and*
4. *$*$-linear bicategories* $(\mathcal{B}, \otimes, \mathbf{1}_X, ⅋, \bot_X, \multimap, \multimapinv)$.

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