Towards a homotopy theory of algebraic weak ω-categories

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Algebraic weak ω-categories

0-cells   1-cells   2-cells   ...

$$\bullet \qquad \bullet \longrightarrow \bullet \qquad \begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd} \qquad \cdots$$

i.e. globular sets = presheaves over

$$\left\langle 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} \cdots \;\middle|\; ss = ts,\ st = tt \right\rangle$$

equipped with compositions & identities

satisfying unit, associativity, & interchange laws

up to "equivalence".

Encode everything as an algebra structure

(for a monad / globular operad).

More on "algebraic"

If we use simplices instead of globes,

$$\bullet\ ,\quad \bullet \longrightarrow \bullet\ ,\quad \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & \bullet \end{tikzcd}\ ,\quad \cdots$$

we can encode the category-like structure as an existence property.

e.g. $\forall\ x \xrightarrow{f} y \xrightarrow{g} z \qquad \exists$
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\wr"] & & z \end{tikzcd}$$

But globes don't have enough faces to do this. $\rightsquigarrow$ (actual) algebraic structure

Idea :
$$\begin{tikzcd} \text{Str-}\omega\text{-Cat} \simeq \mathbf{GSet}^{T} \arrow[rr, "\alpha^{*}"] \arrow[dr] & & \mathbf{GSet}^{L} =: \text{Wk-}\omega\text{-Cat} \arrow[dl] \\ & \mathbf{GSet} & \end{tikzcd}$$

Definition of $\alpha : L \to T$ (Batanin, Leinster)

$\alpha : L \to T$ is the initial cartesian monad over T with contraction.

$$\forall X \in GSet \qquad \begin{tikzcd} LX \arrow[r, "L!"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\text{pullback}"] & L1 \arrow[d, "\alpha_1"] \\ TX \arrow[r, "T!"'] & T1 \end{tikzcd}$$

everything is determined by α_1

$$\forall n \geq 1 \qquad \begin{tikzcd} \partial G^n \arrow[r, "\forall"] \arrow[d, hook] & L1 \arrow[d, "\alpha_1"] \\ G^n \arrow[r, "\forall"'] \arrow[ur, dashed, "\exists \text{ specified}"] & T1 \end{tikzcd}$$

$L1$ = {pasting instructions}

α_1 consists of loops

$T1$ = {pasting schemes}

G^n representable

This encodes the existence part of pasting theorem.

Example: unit law

Intuition: Weak ω-cats have all operations that strict ω-cats have, including those that we usually think of as equations. e.g.

input	output
x	$x \xrightarrow{1} x$
$x \xrightarrow{f} y$	$x \xrightarrow{1} x \xrightarrow{f} y$

$$
\begin{tikzcd}
x \arrow[r, "1"] \arrow[rr, bend right=40, "f"'] & x \arrow[r, "f"] \arrow[d, phantom, "\Downarrow"] & y \\
 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \\
x \arrow[r, "1"'] \arrow[rr, bend left=40, "f"] & x \arrow[r, "f"'] \arrow[u, phantom, "\Downarrow"] & y
\end{tikzcd}
$$

$\vdots$

exhibit this as a coinductive equivalence.

An n-cell $u : a \to b$ is an equivalence if $\exists$

- n-cell $v : b \to a$
- equivalence $(n+1)$-cells $1_a \to vu$ & $1_b \to uv$.

How do we prove things?

Lemma: The uniqueness part of pasting theorem holds up to coinductive equivalence.

Lemma: Coinductive equivalences are closed under all kinds of composition.

Proof strategy: Take the proof of the strict case.
Replace all $=$ by $\sim$.
Compose the resulting sequence of $\sim$'s into single $\sim$.

need something more sophisticated when "equating" e.g. :

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] \arrow[r, phantom, "\Downarrow p"] & y \end{tikzcd} \quad \text{v.s.} \quad \begin{tikzcd} x \arrow[r, "1"] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow p"] & y \end{tikzcd}$$

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Weakly invertible ω-functors

Wk-ω-Cat is an ordinary category.

We need more structure to capture up-to-equivalence stuff.

we are aiming for a left semi-model structure

Definition: An ω-weak equivalence $F : X \longrightarrow Y$ is an ω-functor

algebra morphism

that is

- e.s.o. $\forall y \in Y_0 \ \exists x \in X_0$ s.t. $Fx \sim y$.
- f.f. each induced map

$$X(x, x') \longrightarrow Y(Fx, Fx')$$

is e.s.o. & f.f.

i.e. e.s.o. in every dimension.

"2-out-of-3 property"

Theorem: Given $X \xrightarrow{F} Y \xrightarrow{G} Z$, if any two of F, G, & GF are ω-weak equivalences then so is the third.

ω-dimensional isofibrations

Definition: An ω-equifibration $F : X \to Y$ is an ω-functor s.t.

ω-categorical analogue of isofibrations

$$\begin{tikzcd}
\forall\, x \arrow[r, dashed, "\exists\, \bar{f}", "\sim"'] & \exists\, \bar{y} & \text{in } X \\
Fx \arrow[r, "\forall\, f", "\sim"'] & \forall\, y & \text{in } Y
\end{tikzcd}$$

Theorem: There exists "free coherent equivalence n-cell" E^n s.t. $F : X \to Y$ is an ω-equifibration iff $\forall\, n \geq 1$

coherent equivalence: analogous to adjoint equivalence

free $(n-1)$-cell: C^{n-1}

$$\begin{tikzcd}
C^{n-1} \arrow[r, "\forall"] \arrow[d, hook] & X \arrow[d, "F"] \\
E^n \arrow[r, "\forall"'] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

Path objects?

The remaining problem is constructing path objects $Path(X)$.

free coherent equivalence 1-cell

Intuitively $Path(X) = [E_1, X]$ should do, except that we don't have such a closed structure (yet).

The proof strategy cannot be applied because the strict version is constructed (and verified) by hand.

take the strict case and replace all "$=$" by "$\sim$"

References

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