

Towards a homotopy theory of algebraic weak ω -categories

Yuki Maehara (Tokyo Metropolitan University)

joint work with

Soichiro Fujii (National Institute of Informatics)

Keisuke Hoshino (Kyoto University)

PCT Seminar

Towards a homotopy theory
of algebraic weak ω -categories

Algebraic weak ω -categories

0-cells 1-cells 2-cells ...



i.e. globular sets = presheaves over
 $\langle 0 \xrightleftharpoons[t]{s} 1 \xrightleftharpoons[t]{s} \dots \mid ss=ts, st=tt \rangle$

equipped with compositions & identities

satisfying unit, associativity, & interchange laws

up to "equivalence".

Encode everything as an algebra structure
(for a monad / globular operad).

More on "algebraic"

If we use simplices instead of globes,



we can encode the category-like **structure** as an existence **property**.

e.g. $\forall x \xrightarrow{f} y \xrightarrow{g} z \quad \exists$

A commutative triangle diagram with nodes x , y , and z . There is a blue arrow f from x to y , a blue arrow g from y to z , and a blue arrow h from x to z . A blue arrow labeled 12 points from f to g .

But globes don't have enough faces to do this. $\rightsquigarrow$ **algebraic structure** (actual)

Idea: $\text{Str-}\omega\text{-Cat} \simeq \text{GSet}^T \xrightarrow{\alpha^*} \text{GSet}^L =: \text{Wk-}\omega\text{-Cat}$

A commutative diagram showing the relationship between $\text{Str-}\omega\text{-Cat}$, GSet^T , GSet^L , and GSet . Arrows point from $\text{Str-}\omega\text{-Cat}$ to GSet^T and from GSet^L to GSet . Arrows also point from GSet^T to GSet and from GSet^L to GSet .

Definition of $\alpha: L \rightarrow T$ (Batanin, Leinster)

$\alpha: L \rightarrow T$ is the initial cartesian monad over T with contraction.

$$\forall X \in \mathbf{GSet} \quad \begin{array}{ccc} LX & \xrightarrow{L!} & L1 \\ \alpha_x \downarrow & \text{pullback} & \downarrow \alpha_1 \\ TX & \xrightarrow{T!} & T1 \end{array}$$

everything is determined by α_1

$$\forall n \geq 1 \quad \begin{array}{ccc} \partial G^n & \xrightarrow{\forall} & L1 = \{\text{pasting instructions}\} \\ \downarrow & \nearrow \text{E specified} & \downarrow \alpha_1 \\ G^n & \xrightarrow{\forall} & T1 = \{\text{pasting schemes}\} \end{array}$$

α_1 consists of loops

representable

This encodes the existence part of pasting theorem.

Example: unit law

Intuition: Weak w -cats have **all operations** that strict w -cats have, including those that we usually think of as **equations**. e.g.

input	output
x	$x \xrightarrow{1} x$

$$x \xrightarrow{f} y \quad x \xrightarrow{1} x \xrightarrow{f} y$$

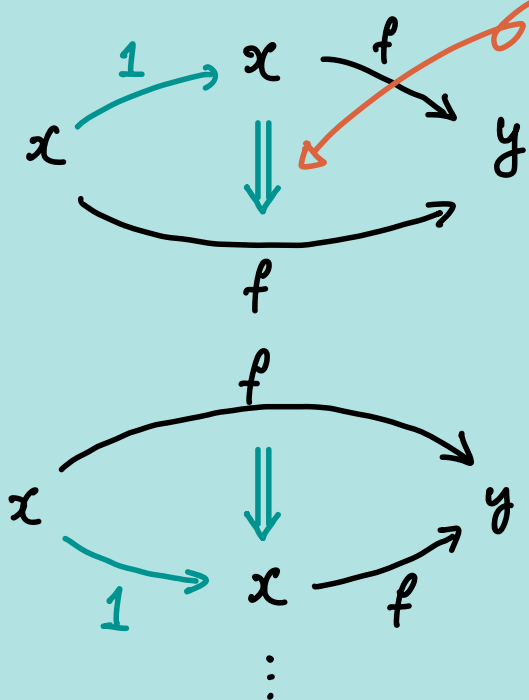


exhibit this as a coinductive equivalence.

An n -cell $u: a \rightarrow b$ is an **equivalence** if $\exists$

- n -cell $v: b \rightarrow a$
- **equivalence** $(n+1)$ -cells
 $1_a \rightarrow vu \neq 1_b \rightarrow uv.$

How do we prove things?

Lemma: The **uniqueness part** of **pasting theorem** holds up to coinductive equivalence.

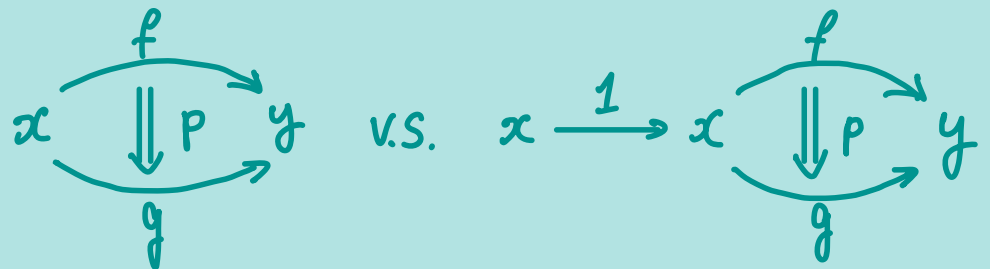
Lemma: Coinductive equivalences are **closed under all kinds of composition**.

Proof strategy: Take the proof of the strict case.

Replace all **=** by **$\sim$** .

Compose the resulting sequence of **$\sim$'s** into single **$\sim$** .

need something more sophisticated
when "equating" e.g. :



Towards a *homotopy theory*
of algebraic weak ω -categories

Weakly invertible ω -functors

$Wk\text{-}\omega\text{-Cat}$ is an ordinary category.

We need more structure to capture **up-to-equivalence** stuff.

└ we are aiming for a left semi-model structure

Definition: An **ω -weak equivalence** $F: X \rightarrow Y$ is an ω -functor that is

algebra morphism

- **e.s.o.** $\forall y \in Y_0 \exists x \in X_0$ s.t. $Fx \sim y$.

- **f.f.** each induced map

$$X(x, x') \longrightarrow Y(Fx, Fx')$$

is e.s.o. & f.f.

i.e. e.s.o. in every dimension.

"2-out-of-3 property"

Theorem: Given $X \xrightarrow{F} Y \xrightarrow{G} Z$, if **any two** of F , G , & GF are ω -weak equivalences then **so is the third**.

ω -dimensional isofibrations

Definition: An ω -equifibration $F: X \rightarrow Y$ is an ω -functor s.t.

ω -categorical
analogue of
isofibrations

$$\begin{array}{ccc} \forall x & \xrightarrow[\sim]{\exists \bar{f}} & \exists \bar{y} & \text{in } X \\ Fx & \xrightarrow[\sim]{\forall f} & \forall y & \text{in } Y \end{array}$$

Theorem: There exists "free coherent equivalence n -cell" E^n s.t.
 $F: X \rightarrow Y$ is an ω -equifibration iff $\forall n \geq 1$

free $(n-1)$ -cell $\curvearrowright$

$$\begin{array}{ccc} C^{n-1} & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists & \downarrow F \\ E^n & \xrightarrow{\forall} & Y \end{array}$$

analogous to
adjoint equivalence

Path objects?

The remaining problem is constructing path objects $\text{Path}(X)$.

Intuitively $\text{Path}(X) = [E_1, X]$ should do, except that we don't have such a closed structure (yet).
free coherent equivalence 1-cell

The proof strategy *cannot* be applied because the strict version is constructed (and verified) by hand.

— take the strict case and
replace all " $=$ " by " $\sim$ "

References

Batanin, Monoidal globular categories as a natural environment for the theory of weak n -categories, *Adv. Math.* 136(1):39-103, 1998.

Leinster, Higher operads, higher categories, volume 298 of London Mathematical Society Lecture Note Series, 2004.

Fujii, Hoshino, M., Weakly invertible cells in a weak w -category, *High. Struct.* 8(2):386-415, 2024.

Fujii, Hoshino, M., w -weak equivalences between weak w -categories, *Adv. Math.* 480:110490, 2025.

Fujii, Hoshino, M., w -equifibrations between strict and weak w -categories, *arXiv*:2511.09849, 2025.