

Contractions via moduli of sheaves

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Plan of the talk

Introduction

Moduli problems

- Classification questions

- Moduli functors

- Stacks

Main results

- Good moduli spaces

- Examples

Applications



Andres Fernandez Herrero

Motivating question

X – smooth projective variety over a field k

$j: Y \rightarrow X$ – closed subvariety

$\pi: Y \rightarrow B$ – surjective morphism

Question

When does there exist a proper algebraic space $\overline{X}$ with a surjective $f: X \rightarrow \overline{X}$ such that

- ▶ $f(Y) = B \hookrightarrow \overline{X}$, and
- ▶ $X \setminus Y \rightarrow \overline{X} \setminus B$ is an isomorphism?

Classifying

- ▶ Groups: finitely generated abelian; finite; Lie; reductive
- ▶ Lie algebras: semisimple
- ▶ Varieties: curves; K3 surfaces; homogeneous spaces; K-stable Fano
- ▶ Sheaves on varieties: vector bundles, coherent sheaves, Bridgeland stable complexes
- ▶ Representations: of Lie algebras, of quivers
- ▶ Objects in abelian categories

Families

Question

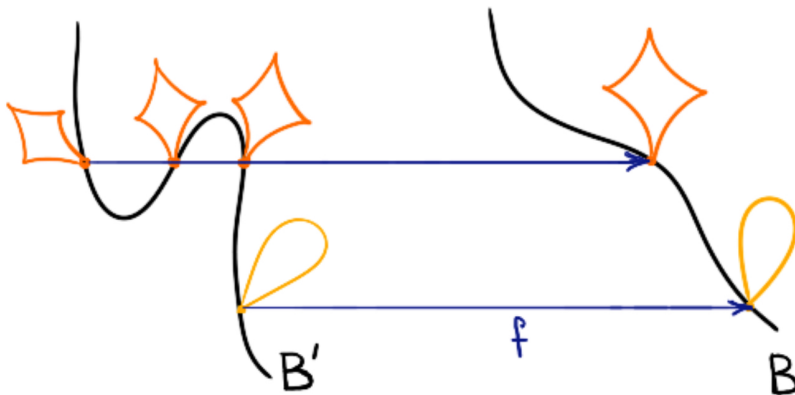
How to say when two objects
“differ slightly”?

Approach

Consider families



Pullback of families



Moduli functor

Definition

- *Moduli problem* = $\mathcal{M}: \underline{Sch}^{\text{op}} \rightarrow \underline{Set}$ such that $\mathcal{M}(\text{Spec } k) = \text{set of isoclasses of objects}$
- *Fine moduli space* = $\underline{M} \in \underline{Sch} + \underline{\mathcal{M}} \cong \underline{\text{Hom}}(_, \underline{M})$

Definition

Family = $F \in \mathcal{M}(B)$

Univ. family $\mathcal{E} \in \mathcal{M}(M) = \text{Hom}(M, M)$

Example 1: functor of points *aka the repres. functor*

X smooth projective variety over a field k

$$T \in \mathcal{S}ch_k \rightsquigarrow \underline{\mathcal{M}(T)} := \text{Hom}(T, X)$$

$$\underline{k} \subset \underline{K} \text{ field extension} \rightsquigarrow \underline{\mathcal{M}(\text{Spec } K)} = X(K) = K\text{-points}$$

X is the fine moduli space for $\mathcal{M}$

Example 2: functor of several points (aka Hilbert scheme of points)

X smooth projective variety over a field k

$\ell > 0$ natural number

$$T \in \mathcal{S}ch_k \rightsquigarrow \mathcal{M}(T) := \left\{ Z \subset X \times T \mid \begin{array}{l} Z \text{ is } T\text{-flat;} \\ \forall t \in T: Z|_{X \times t} \text{ length } \ell \text{ subscheme} \end{array} \right\}$$

Theorem

There is a fine moduli space of $\mathcal{M}$, denoted $\mathrm{Hilb}^\ell(X)$

Problem

Moduli functors are rarely representable

Solution 1

Throw out the non-schematic locus (stability, GIT)

Solution 2

Expand the notion of moduli functor

Why stacks?

Solution 2

Ask the moduli functor remember automorphisms $\rightsquigarrow$ stacks

Advantages

- ▶ Works when GIT cannot be applied, when we don't know enough about the structure
- ▶ Intrinsic, or “coordinate-free”

Definition

Definition

Groupoid = category, all morphisms are isomorphisms

$\mathcal{G}pd$ = 2-category of groupoids

Definition

$Stack = \mathcal{M}: \mathcal{S}ch_k^{op} \rightarrow \mathcal{G}pd + \text{gluing condition}$

Definition

$\mathcal{M}$ algebraic =

$\exists$ smooth surjection $U \rightarrow \mathcal{M}$ from some scheme U ,
 and $\mathcal{M} \rightarrow \mathcal{M} \times_k \mathcal{M}$ is representable

Example 0: $B\mathbb{G}_m$

$\mathbb{G}_m := \text{mult. group of } k$

$$\mathcal{M}(\text{Spec } k) = \{ \underset{\substack{\uparrow \\ *}}{1} \text{ obj}, \mathbb{G}_m(\text{Spec } k) = \text{Aut}(*) \}$$

k a field

$T \in \text{Sch}_k \rightsquigarrow \mathcal{M}(T) := \{\mathbb{G}_m\text{-torsors over } T\}$

Morphisms in $\mathcal{M}(T) = \text{isomorphisms}$

$$\begin{array}{ccc} \mathbb{G}_m \times P & \rightarrow & P \\ \downarrow & \swarrow & \\ T & & \end{array}$$

Definition

Denote this stack $\mathcal{M}$ by $B\mathbb{G}_m$

Example 1: ideal sheaves of one point

X smooth projective variety over a field k

$$T \in \mathcal{S}ch_k \rightsquigarrow \mathcal{M}(T) := \left\{ \mathcal{I}_Z \text{ ideal sheaf} \mid \begin{array}{l} Z \subset X \times T \text{ a } T\text{-flat subscheme;} \\ \forall t \in T: Z|_{X \times t} \text{ length 1 subscheme} \end{array} \right\}$$

Morphisms in $\mathcal{M}(T)$ = isomorphisms of sheaves (closed pt.)

Lemma

If $\dim X > 1$, then $\mathcal{M} \cong X \times B\mathbb{G}_m$

Why? Smooth, $\dim \geq 2 \Rightarrow$ if $\mathcal{I}_p \cong \mathcal{I}_q$, then $p=q$.

Example 2: ideal sheaves of several points

X smooth projective variety over a field k

$\ell > 0$ natural number

$T \in \text{Sch}_k \rightsquigarrow$

$$\mathcal{M}(T) := \left\{ \mathcal{I}_Z \text{ ideal sheaf} \mid \begin{array}{l} Z \subset X \times T \text{ a } T\text{-flat subscheme;} \\ \forall t \in T: Z|_{X \times t} \text{ length } \ell \text{ subscheme} \end{array} \right\}$$

Morphisms in $\mathcal{M}(T) =$ isomorphisms of sheaves

Lemma

If $\dim X > 1$, then $\mathcal{M} \cong \text{Hilb}^\ell(X) \times B\mathbb{G}_m$

Example 3: stack $\mathcal{U}_{\{\mathbb{P}^1\},1}$

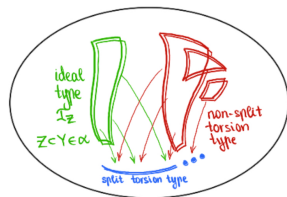
Setup

- ▶ $i : \mathbb{P}^1 \hookrightarrow X$ rigid smooth rational curve
(i.e. $\{\mathbb{P}^1\} \subset \text{Hilb}(X)$ is open and closed)
- ▶ $\mathcal{N}_{\mathbb{P}^1/X}$ splits into line bundles of degree ≤ 0

Points of $\mathcal{U}_{\{\mathbb{P}^1\},1}$ of two types:

- ▶ $F \cong \mathcal{I}_x$ for $x \in X$
- ▶ F that fits into SES

$$0 \rightarrow \underline{i_*(\mathcal{O}_{\mathbb{P}^1}(-1))} \rightarrow F \rightarrow \mathcal{I}_{\mathbb{P}^1} \rightarrow 0$$



Good moduli spaces

Definition

Good moduli space (gms) for $\mathcal{M} = \text{qcqs morphism } \varphi: \mathcal{M} \rightarrow M$ to an algebraic space M such that

- ▶ $\varphi_*: \text{QCoh } \mathcal{M} \rightarrow \text{QCoh } M$ exact
- ▶ $\mathcal{O}_M \rightarrow \varphi_* \mathcal{O}_{\mathcal{M}}$ isomorphism

Example

G (linearly reductive) acts on variety X ,
 $X \rightarrow Y$ good quotient $\Rightarrow [X/G] \rightarrow Y$ gms

↑
quotient stack

Thrm [Alper]
 φ is initial among
mms $\{ \mathcal{M} \rightarrow T \}$,
 T : alg. space

Ex. $\mathbb{A}^n_{\mathbb{G}m} \hookrightarrow *$
 $B\mathbb{G}m \rightarrow *$ is a gms

Existence criterion [Alper–Halpern-Leistner–Heinloth '18-'23]

$(\mathcal{Coh}(X))_{\det \cong 0}$ is like this)

$\mathcal{M}$ algebraic, finite type, affine stabilisers, affine diagonal

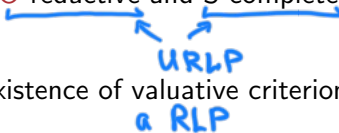
Theorem

$\mathcal{M}$ has a separated gms $\mathcal{M} \iff$

$\mathcal{M}$ is locally linearly reductive, Θ -reductive and S-complete

Theorem

$\mathcal{M}$ is proper $\iff$ $\mathcal{M}$ satisfies existence of valuative criterion for properness



Examples

Example 0

$$B\mathbb{G}_m \rightarrow \operatorname{Spec} k \text{ gms}$$

Ex 0': If X is a scheme,
then $X \rightarrow X$ is a gms.

Example 1: ideal sheaves of 1 point

$$\mathcal{H} \rightarrow X$$

$$X \times B\mathbb{G}_m \rightarrow X \text{ gms}$$

Example 2: ideal sheaves of ℓ points

$$\operatorname{Hilb}^\ell(X) \times B\mathbb{G}_m \rightarrow \operatorname{Hilb}^\ell(X) \text{ gms}$$

Examples 3 and 4

Theorem A [FH, M '25]

$\mathcal{U}_{\alpha,\ell}$ has a gms, denoted $\mathrm{Hilb}^\ell(X)_\alpha$

Example 4: stack $\mathcal{U}_{\{\mathbb{P}^1\},1}^\circ \subset \mathcal{U}_{\{\mathbb{P}^1\},1}$

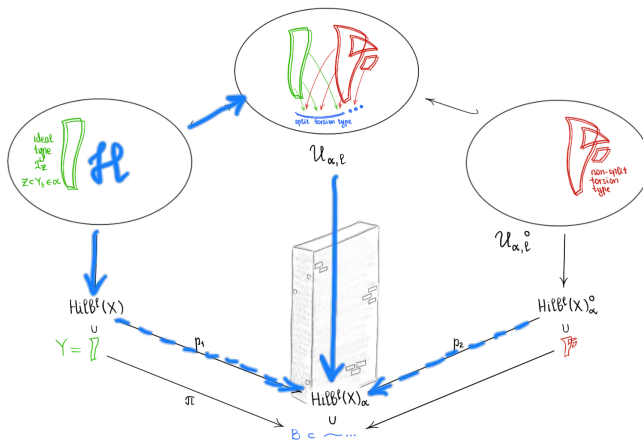
Take sheaves F of two types:

- ▶ $F \cong \mathcal{I}_x$ for $x \in X \setminus \mathbb{P}^1$
- ▶ fits into non-split SES $0 \rightarrow i_*(\mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow F \rightarrow \mathcal{I}_W \rightarrow 0$

Theorem B [FH, M '25]

$\mathcal{U}_{\alpha,\ell}^\circ$ has a gms, denoted $\mathrm{Hilb}^\ell(X)_\alpha^\circ$

Examples 3 and 4



Contracting rational curves

Known results

- ▶ [Reid '83] $\dim X = 3$,
 $\mathcal{N}_{\mathbb{P}^1/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$
 $\implies \mathbb{P}^1$ contracts
- ▶ [Jiménez '92] W – smooth curve of genus g – contracts
 $\iff \exists$ l.b. $\mathcal{L}$ on a nbhd of W in X s.t.:
 $\deg(\mathcal{L}|_W) \leq g - 2$ and $H^1(\hat{X}, \hat{\mathcal{L}})$ is finite-dimensional

Corollary of Theorems A and B

Note: $\mathcal{N}_{\mathbb{P}^1/X}$ splits as $\bigoplus_i \mathcal{O}(n_i)$. If all $n_i \leq 0$, then:

$\mathbb{P}^1$ contracts in $\overline{X}$ and gets replaced by $\mathbb{P}H^1(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1/X}(-1))$ in X° .

Abel-Jacobi contractions

$$\dim X = 2$$

$\alpha \subset \text{Hilb}(X)$ open and closed subfamily of some curves

$\mathcal{C} \rightarrow \alpha$ universal curve

Abel-Jacobi morphism $AJ_\ell: \text{Hilb}^\ell(\mathcal{C}/\alpha) \rightarrow \overline{\text{Pic}}(\mathcal{C}/\alpha)_{\chi(\mathcal{O})-\ell}$

$$\begin{array}{ccccc}
 & \text{Hilb}^\ell(\mathcal{C}/\alpha) & \xrightarrow{\quad} & \text{Hilb}^\ell(X) & \\
 & \downarrow & & \downarrow & \\
 \text{Pic}(\mathcal{C}/\alpha)_{\chi(\mathcal{O})-\ell} & \xleftarrow{\quad} & H_\ell & \xrightarrow{\quad} & \text{Hilb}^\ell(X)_\alpha
 \end{array}$$

AJ_ℓ (diagonal arrow from $\text{Hilb}^\ell(\mathcal{C}/\alpha)$ to $\text{Pic}(\mathcal{C}/\alpha)_{\chi(\mathcal{O})-\ell}$)

A Mukai flop

$X \rightarrow \mathbb{P}^1$ – elliptic K3 with two singular fibers X_a, X_b of type I_3

$X_a = C_1 \cup C_2 \cup C_3$ – cycle of three $\mathbb{P}^1$'s

$\alpha = \{C_1, C_2, C_3\}$ – isolated points in $\text{Hilb}(X)$

$\ell \geq 2 \implies$ condition $(\dagger)$ is satisfied

Corollary of Theorems A and B

- ▶ $\text{Hilb}^\ell(X) \rightarrow \text{Hilb}^\ell(X)_\alpha$ is a contraction of $\bigsqcup \text{Hilb}^\ell(C_i)$
- ▶ $\text{Hilb}^\ell(X)_\alpha^\circ$ is a Mukai flop

$\text{Hilb}^2(X)_\alpha$ is not...

- ▶ ...a scheme
- ▶ ...a Bridgeland moduli space (all such admit a strictly nef l.b.)

$\mathbb{P}^1$ -fibration surgeries, $\ell = 1$

X – smooth projective variety

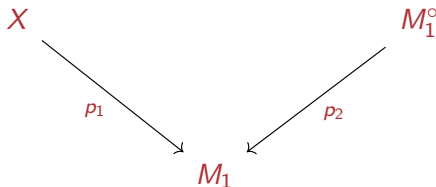
$Y \subset X$ – subvariety with a $\mathbb{P}^1$ -fibration $Y \rightarrow B$ s.t.

B corresponds to an open and closed subscheme $\alpha \subset \text{Hilb}(X)$

$\mathcal{N}_{Y/X}$ is fiberwise non-positive

Corollary of Theorems A and B

We have a surgery diagram



DK conjecture

X – smooth projective variety

$X \supset Y \rightarrow B$ – $\mathbb{P}^1$ -fibration

$B \longleftrightarrow \alpha \subset \text{Hilb}(X)$

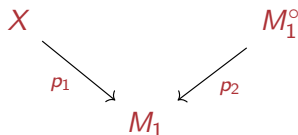
Assume M_1° is smooth

$\mathcal{N}_{Y/X}$ is fiberwise non-positive

DK conjecture [Kawamata '18, Conj. 1.2]

If $\forall b \in B: \omega_X \cdot Y_b \leq 0$, then

$\exists$ an embedding of an admissible subcategory $D^b(M_1^\circ) \rightarrow D^b(X)$.



DK conjecture

Theorem [Bridgeland '01]

M – complex projective threefold with terminal singularities,

$X \rightarrow M_1 \leftarrow M_1^\circ$ – crepant resolutions.

Then $D^b(M_1^\circ) \cong D^b(X)$.

DK conjecture

Theorem [Bridgeland '01]

M – complex projective threefold with terminal singularities,
 $X \rightarrow M_1 \leftarrow M_1^\circ$ – crepant resolutions.
 Then $D^b(M_1^\circ) \cong D^b(X)$.

Theorem [FH, M '25]

Verified the DK conjecture in the above setting.

Our assumptions $\implies \omega_X \cdot Y_b = -2, -1, 0$.

- ▶ Case $\omega_X \cdot Y_b = -2$: $M_1^\circ = \emptyset$
- ▶ Case $\omega_X \cdot Y_b = -1$: $M_1^\circ =$ Fujiki-Nakano contraction
- ▶ Case $\omega_X \cdot Y_b = 0$: $D^b(M_1^\circ) \cong D^b(X)$

Thank you!