From concept mining to categorical nuclei and tight completions

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ongoing joint work with
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Pacific Category Theory Seminar
24 April 26

Outline

Background: Decompositions

New problem: Concept mining

Approach: Nuclear adjunctions

Old problem: Tight completions

Solution: Categorical cuts and concepts

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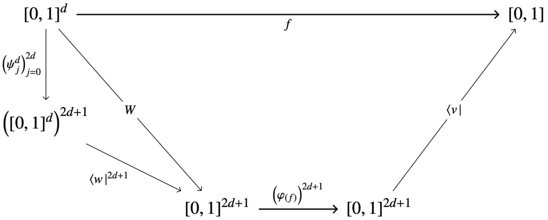
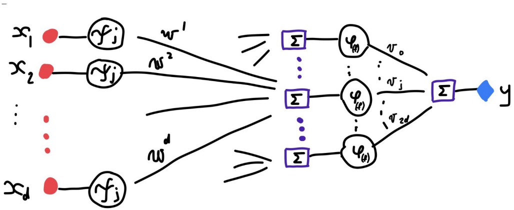
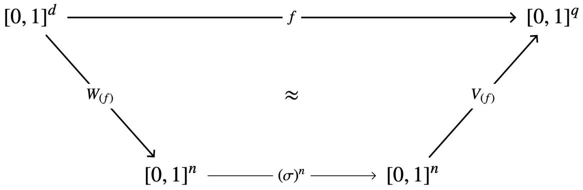
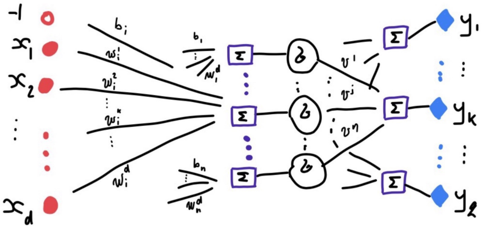
Kolmogorov-Arnold Decomposition



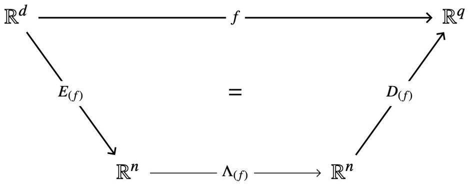


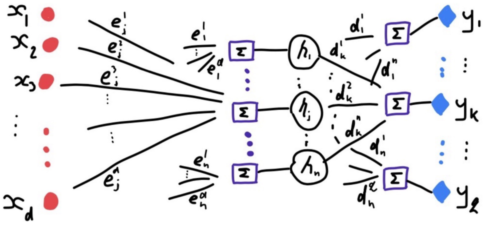
Neural **Approximation**





Concept (Singular Value) Decomposition





Outline

Background: Decompositions

New problem: Concept mining

From contexts to concepts

Formal Concept Analysis

Latent Semantic Analysis

Waves and particles of meaning

Categorical concept analysis

Approach: Nuclear adjunctions

Old problem: Tight completions

Web semantics = Concept spaces

- concept = mixture of words

Web semantics = Concept spaces

- concept = mixture of words

$\Downarrow$

- word = mixture of concepts

Web semantics = Concept spaces

- concept = mixture of words

$\Downarrow$

- word $\leftsquigarrow$ mixture of contexts

Web semantics = Concept spaces

- concept = mixture of words

$\Downarrow$

- word $\leftsquigarrow$ mixture of contexts

$\Downarrow$

- concept $\leftsquigarrow$ context invariant

Context snapshot

Φ	Utamaro	Kagemusha	Totoro	Paprika
Alice	$\star\star\star\star$	$\star\star$	$\star\star\star\star$	$\star$
Bob	$\star\star$	$\star\star\star$	$\star\star$	
Carol	$\star\star$	$\star$	$\star\star\star\star\star$	
Dave	$\star$	$\star\star\star\star$	$\star\star\star$	
Ed		$\star\star$	$\star\star\star\star\star$	$\star\star$

$$\Phi : \mathcal{A} \times \mathcal{B} \to \{1, 2, \ldots, 5\}$$

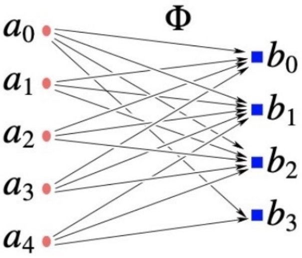
Context snapshot induces concept space

domain	$\mathcal{A}$	$\mathcal{B}$	R	Φ_{ab}
user preference	items	users	$\{1,\ldots,5\}$	rating
text analysis	documents	terms	$\mathbb{N}$	occurrence
topic search	authorities	hubs	$\mathbb{N}$	hyperlinks
measurement	instances	quantities	$\mathbb{R}$	outcome
concept analysys	objects	attributes	$\{0,1\}$	property
elections	candidates	voters	$\{1,\ldots,n\}$	preference
market	producers	consumers	$\mathbb{Z}$	deliveries
digital images	positions	pixels	$[0,1]$	intensity

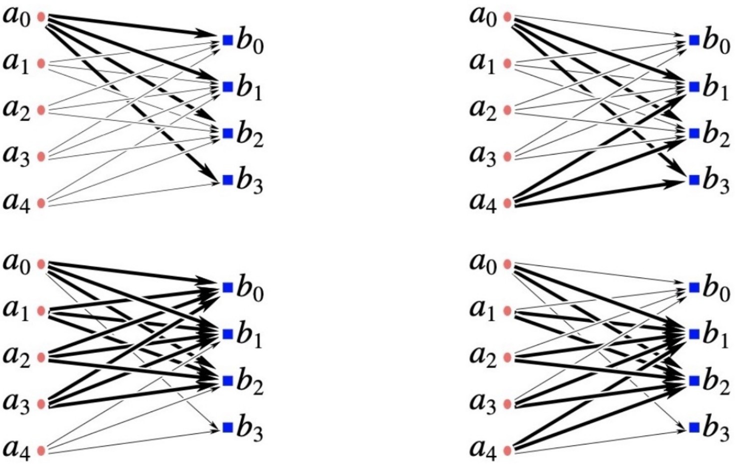
Formal Concept Analysis

Φ	Utamaro	Kagemusha	Totoro	Paprika
Alice	★	★	★	★
Bob	★	★	★	
Carol	★	★	★	
Dave	★	★	★	
Ed		★	★	★

Formal Context

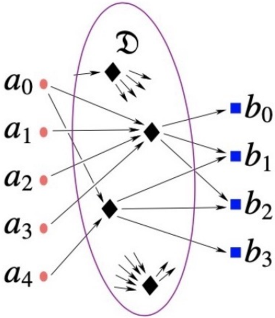


Formal Concepts

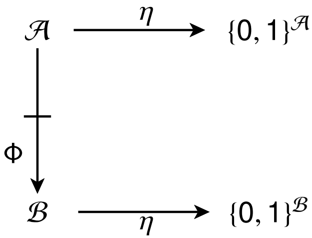


Concept lattice

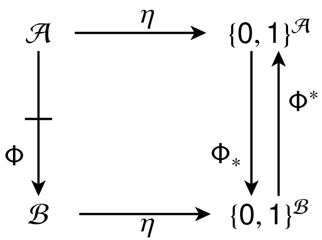
Formal Concept Analysis (FCA)



$\bigcup$-completions



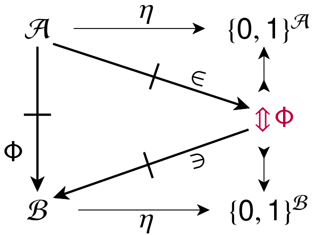
$\bigcup$-completions $\rightsquigarrow$ Galois connections



$$\Phi_* X = \bigcap_{x \in X} x\Phi \qquad \text{where } x\Phi = \{y \in \mathcal{B} \mid x\Phi y\}$$

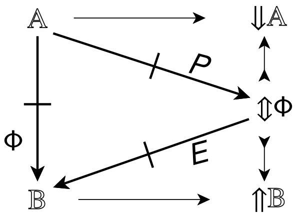
$$\Phi^* Y = \bigcap_{y \in Y} \Phi y \qquad \text{where } \Phi y = \{x \in \mathcal{A} \mid x\Phi y\}$$

Tight $\bigvee\bigwedge$-completion $\rightsquigarrow$ concept lattice



$$a\Phi b \;=\; \bigvee_{c\in\Updownarrow\Phi} a \in c \wedge c \ni b$$

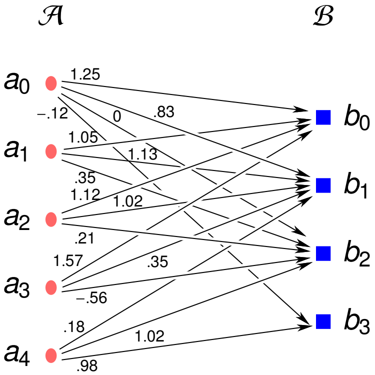
Tight $\bigvee\bigwedge$-completion $\rightsquigarrow$ concept lattice



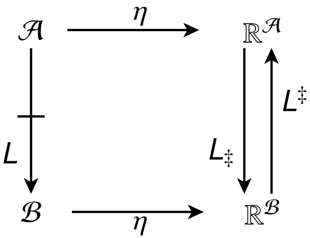
Latent Semantic Analysis

	a_0	a_1	a_2	a_3	a_4
b_0	1.25	1.05	1.12	1.57	
b_1	.83	1.13	1.02	.35	.18
b_2	0	.35	.21	-.56	1.02
b_3	-.12				.98

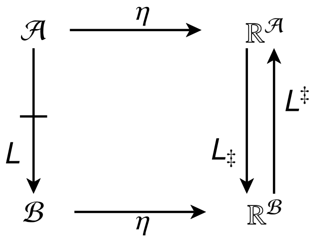
Latent Semantic Analysis



Vector spaces as completions



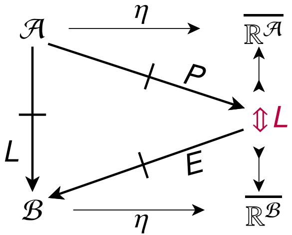
Matrix completion $\rightsquigarrow$ adjoint operators



$$L_{\ddagger}\alpha = \left(\sum_{j=1}^{n} L_{ij}\alpha_j\right)_{i=1}^{m}$$

$$L^{\ddagger}\beta = \left(\sum_{i=1}^{m} \beta_i L_{ij}\right)_{j=1}^{n}$$

Eigenspaces ⇝ tight completion ⇝ concept space



$$L_{ij} = \sum_{\gamma \in \Updownarrow L} \lambda_\gamma \cdot E_{i\gamma} \cdot P_{\gamma j}$$

Spectrum $\rightsquigarrow$ dominant concepts

Concepts are the eigenspaces of $L_{\ddagger}L^{\ddagger}$ and $L^{\ddagger}L_{\ddagger}$, because

$$\gamma_{\mathcal{A}} = L^{\ddagger}\gamma_{\mathcal{B}} \quad \wedge \quad L_{\ddagger}\gamma_{\mathcal{A}} = \gamma_{\mathcal{B}}$$

$$\Updownarrow$$

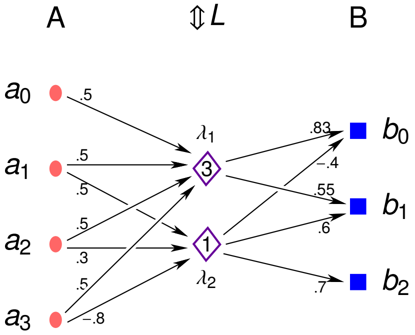
$$\gamma_{\mathcal{A}} = L^{\ddagger}L_{\ddagger}\gamma_{\mathcal{A}} \quad \wedge \quad L_{\ddagger}L^{\ddagger}\gamma_{\mathcal{B}} = \gamma_{\mathcal{B}}$$

Singular Value Decomposition (SVD)

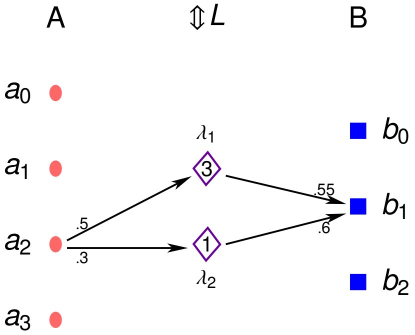
$$\begin{pmatrix} 1.25 & 1.05 & 1.12 & 1.57 \\ .83 & 1.13 & 1.02 & .35 \\ 0 & .35 & .21 & -.56 \end{pmatrix} =$$

$$\begin{pmatrix} .83 & -.4 \\ .55 & .6 \\ 0 & .7 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} .5 & .5 & .5 & .5 \\ 0 & .5 & .3 & -.8 \end{pmatrix}$$

Latent concepts

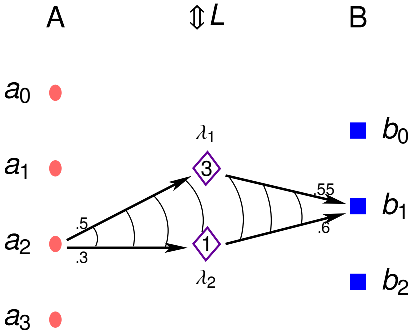


Latent concepts



$$L_{21} = 3(.5 \times .55) + (.3 \times .6)$$

Latent concepts are **waves** of meaning



$$L_{21} = 3(.5 \times .55) + (.3 \times .6)$$

Wave mechanics = Interference.

Matrix algebra = Noninterference

$$\Pr(j|i) \quad = \quad \sum_{\gamma \in \Updownarrow L} \lambda_\gamma \cdot \Pr(\gamma|i) \cdot \Pr(j|\gamma)$$

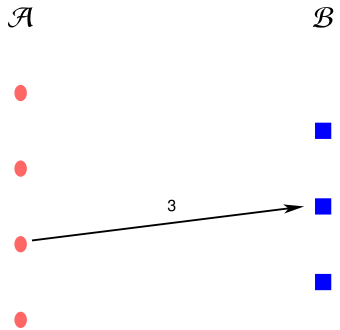
Matrix algebra = Noninterference

$$\Pr(j|i) \quad = \quad \sum_{\gamma \in \Updownarrow L} \lambda_\gamma \cdot \Pr(\gamma|i) \cdot \Pr(j|\gamma)$$

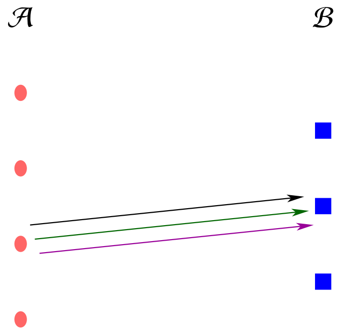
$\Downarrow$

all correlations amplified

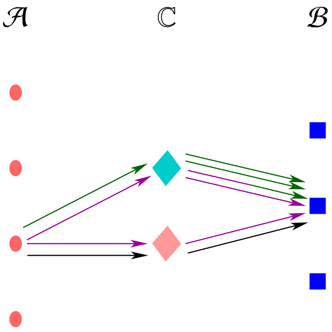
Data **counts** mask interferences



Data **sets** record interferences



Data sets record **concept** interferences



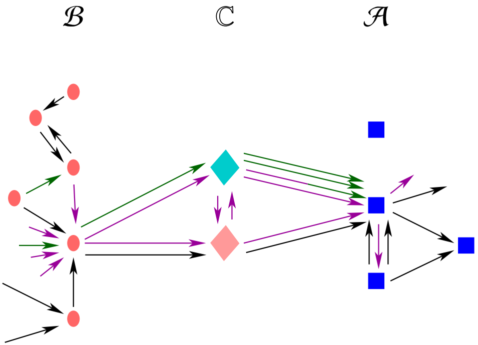
$$\uparrow \;\mapsfrom\; \langle \uparrow, \uparrow \rangle$$

$$\uparrow \;\mapsfrom\; \langle \uparrow, \uparrow\uparrow\uparrow \rangle$$

$$\uparrow \;\mapsfrom\; \langle \uparrow, \uparrow\uparrow \rangle, \langle \uparrow, \uparrow \rangle$$

Categorical matrices ⇝ concept interferences

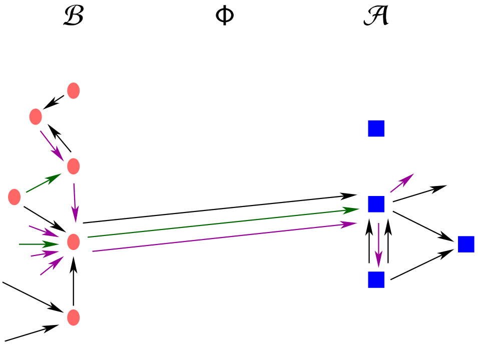
(a.k.a. distributors, profunctors, bimodules)



Previously mined concepts act
on currently mined concepts

Categorical matrices

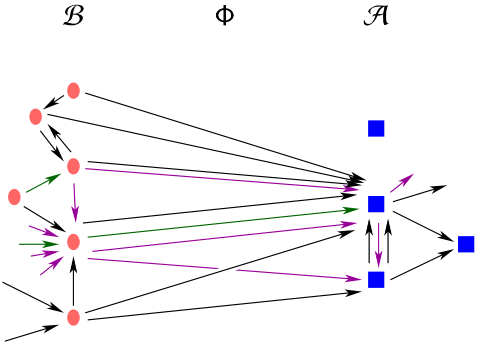
(a.k.a. distributors, profunctors, bimodules)



Concept interactions

Categorical matrices

(a.k.a. distributors, profunctors, bimodules)



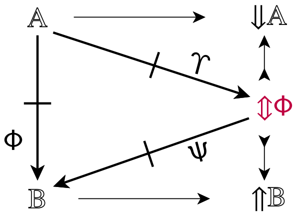
Concept interactions
record data dependencies.

$\varinjlim$, $\varprojlim$-completions ⇝ adjoint functors

$$
\begin{array}{ccc}
\mathbb{A} & \longrightarrow & \mathsf{Set}^{\mathbb{A}^o} \\
\Phi \big\downarrow\!\!\!\!\!+ & & \Phi^* \big\downarrow \big\uparrow \Phi_* \\
\mathbb{B} & \longrightarrow & \left(\mathsf{Set}^{\mathbb{B}}\right)^o
\end{array}
$$

$$
\begin{aligned}
\Phi^* \overleftarrow{\alpha} &= \varinjlim_{x} \Phi_x \overleftarrow{\alpha}_x \\
\Phi_* \overrightarrow{\beta} &= \varprojlim_{y} \overrightarrow{\beta}_y \Phi_y
\end{aligned}
$$

Tight bicompletion ⇝ concept category



$$\langle a|\Phi|b\rangle \quad = \quad \int_{\gamma\in C} \langle a|\Upsilon|\gamma\rangle \times \langle\gamma|\Psi|b\rangle$$

Outline

Background: Decompositions

New problem: Concept mining

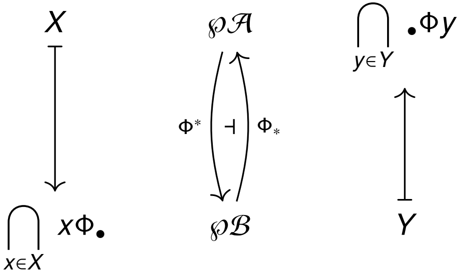
Approach: Nuclear adjunctions

Old problem: Tight completions

Solution: Categorical cuts and concepts

Matrices $\rightsquigarrow$ adjunctions

$$\frac{\dfrac{\Phi : \mathcal{A}\times\mathcal{B}\to 2}{\Phi_\bullet : \mathcal{A}\to 2^{\mathcal{B}} \cong \wp\mathcal{B} \qquad\qquad {}_\bullet\Phi : \mathcal{B}\to 2^{\mathcal{A}} \cong \wp\mathcal{A}}}{\Phi^* : \wp\mathcal{A} \longrightarrow \wp\mathcal{B} \qquad\qquad \Phi_* : \wp\mathcal{B}\longrightarrow \wp\mathcal{A}}$$



Matrices $\rightsquigarrow$ adjunctions

$$\frac{\Phi : \mathbb{A}^o \times \mathbb{B} \to \mathsf{Set}}{\dfrac{\Phi_\bullet : \mathbb{A} \to \left(\mathsf{Set}^{\mathbb{B}}\right)^o \simeq \Uparrow\mathbb{B} \qquad\qquad {}_\bullet\Phi : \mathbb{B} \to \mathsf{Set}^{\mathbb{A}^o} \simeq \Downarrow\mathbb{A}}{\Phi^* : \Downarrow\mathbb{A} \longrightarrow \Uparrow\mathbb{B} \qquad\qquad \Phi_* : \Uparrow\mathbb{B} \longrightarrow \Downarrow\mathbb{A}}}$$

$$\begin{array}{ccccc}
\mathbb{X} \xrightarrow{X} \mathbb{A} & & \Downarrow\mathbb{A} & & \varprojlim\left(\mathbb{Y} \xrightarrow{Y} \mathbb{B} \xrightarrow{{}_\bullet\Phi} \Downarrow\mathbb{A}\right) \\
\big\downarrow & & \Phi^* \; \dashv \; \Phi_* & & \big\uparrow \\
\varinjlim\left(\mathbb{X} \xrightarrow{X} \mathbb{A} \xrightarrow{\Phi_\bullet} \Uparrow\mathbb{B}\right) & & \Uparrow\mathbb{B} & & \mathbb{Y} \xrightarrow{Y} \mathbb{B}
\end{array}$$

Assumption

Categories $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ are[1]

- skeletal
- absolute completions

[1] WLOG: Any category has a skeletal absolute completion.

Assumption

Categories $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$ are[1]

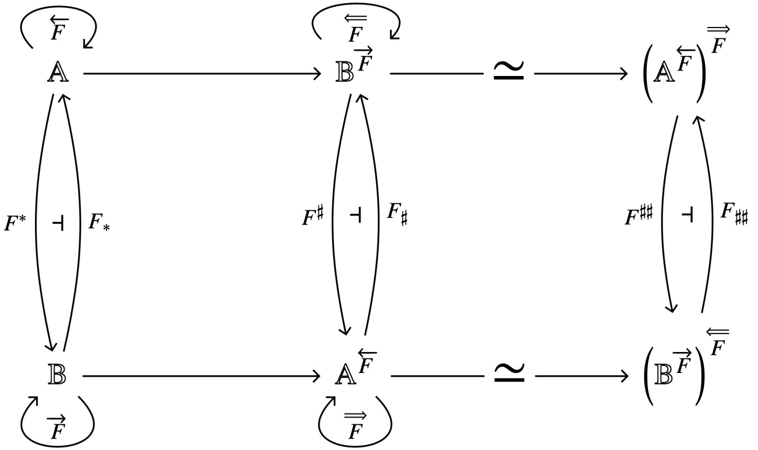
- skeletal
- absolute completions

Lemma

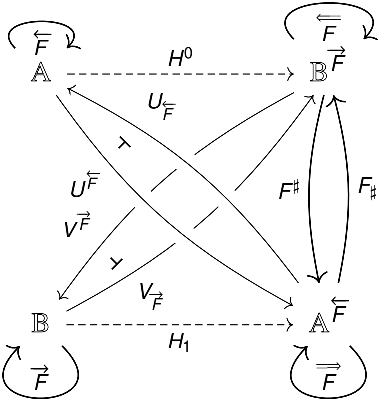
$$\mathbb{A} \simeq \mathbb{B} \quad \Longrightarrow \quad \mathbb{A} \cong \mathbb{B}$$

[1] WLOG: Any category has a skeletal absolute completion.

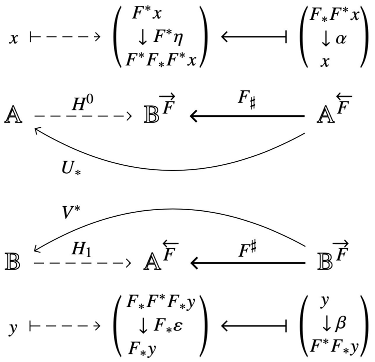
Nucleus theorem



Proof sketch

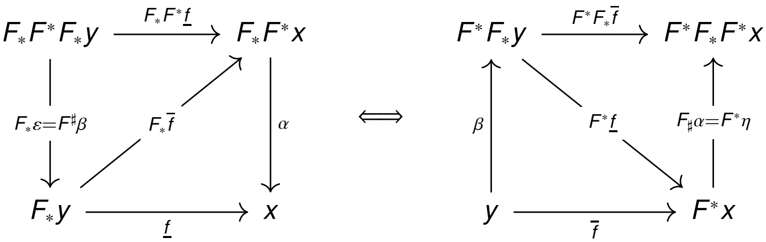


Proof sketch



Proof that the nucleus is an adjunction

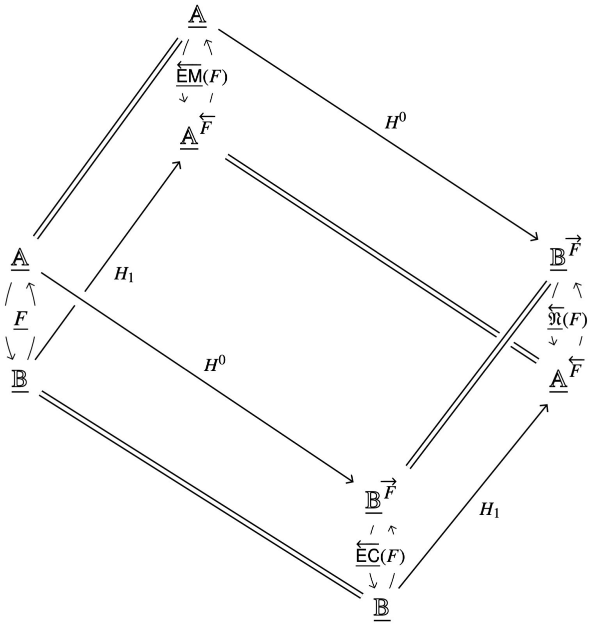
$$\mathbb{A}^{\overleftarrow{F}}\left(F^\sharp\beta, \alpha\right) \qquad \cong \qquad \mathbb{B}^{\overrightarrow{F}}\left(\beta, F_\sharp\alpha\right)$$



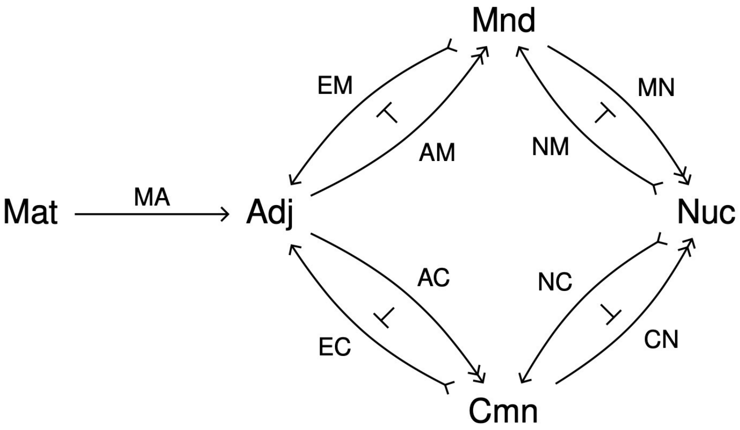
Proof that the nucleus is (co)monadic

https://arxiv.org/abs/2004.07353

Nucleus as an intersection of resolutions

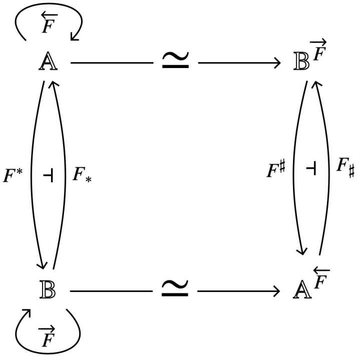


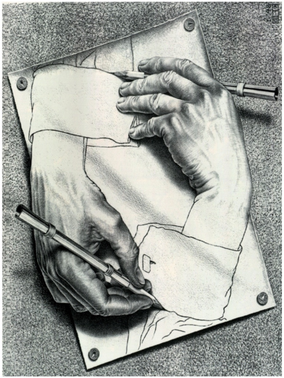
Nuclear data mining workflow



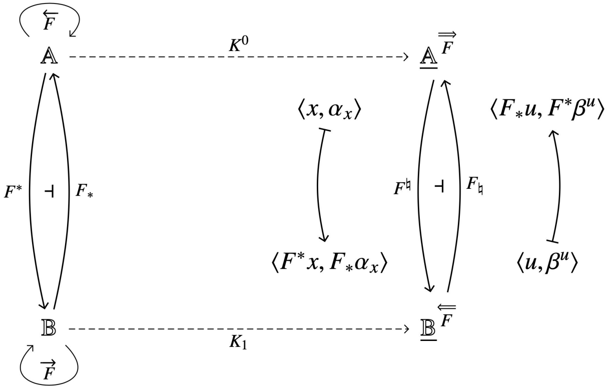
Nuclear adjunction = monadic + comonadic

[M. Barr, Batelle 1968]





Simple nucleus



Coalgebras = coalgebras over algebras

$$\mathbb{B}^{\overrightarrow{F}} \cong \left(\mathbb{A}^{\overleftarrow{F}}\right)^{\overset{\Longrightarrow}{F}} \cong \mathbb{A}^{\overset{\Longrightarrow}{F}}$$

Algebras = algebras over coalgebras

$$\mathbb{A}^{\overleftarrow{F}} \cong \left(\mathbb{B}^{\overrightarrow{F}}\right)^{\overset{\Longleftarrow}{F}} \cong \mathbb{B}^{\overset{\Longleftarrow}{F}}$$

Concepts as invariants

$$|\underline{\mathbb{A}}^{\overrightarrow{F}}| = \coprod_{x\in|\underline{\mathbb{A}}|} \left\{ \alpha_x \in \underline{\mathbb{B}}(F^*x, F^*x) \;\middle|\; \begin{array}{ccccc} F^*x & & F_*F^*x & \xrightarrow{\qquad} \!\!\!\!\rightarrow & x \\ \downarrow{\scriptstyle \alpha_x} & \searrow{\scriptstyle \alpha_x} & & \searrow{\scriptstyle F_*\alpha_x} & \downarrow{\scriptstyle \tilde{\alpha}_x} \\ F^*x & \xrightarrow{\ \alpha_x\ } & F^*x & & F_*F^*x \end{array} \right\}$$

$$\underline{\mathbb{A}}^{\overrightarrow{F}}(\alpha_x, \gamma_z) = \left\{ f \in \underline{\mathbb{A}}(x,z) \;\middle|\; \begin{array}{ccc} F^*x & \xrightarrow{F^*f} & F^*z \\ \downarrow{\scriptstyle \alpha_x} & & \downarrow{\scriptstyle \gamma_z} \\ F^*x & \xrightarrow{F^*f} & F^*z \end{array} \right\}$$

$$|\underline{\mathbb{B}}^{\overleftarrow{F}}| = \coprod_{u\in|\underline{\mathbb{B}}|} \left\{ \beta^u \in \underline{\mathbb{A}}(F_*u, F_*u) \;\middle|\; \begin{array}{ccccc} F_*u & & F^*F_*u & \xrightarrow{\ \tilde{\beta}^u\ } \!\!\!\!\rightarrow & u \\ \downarrow{\scriptstyle \beta^u} & \searrow{\scriptstyle \beta^u} & & \searrow{\scriptstyle F^*\beta^u} & \downarrow \\ F_*u & \xrightarrow{\ \beta^u\ } & F_*u & & F^*F_*u \end{array} \right\}$$

$$\underline{\mathbb{B}}^{\overleftarrow{F}}(\beta^u, \delta^w) = \left\{ g \in \underline{\mathbb{B}}(u,w) \;\middle|\; \begin{array}{ccc} F_*u & \xrightarrow{F_*g} & F_*w \\ \downarrow{\scriptstyle \beta^u} & & \downarrow{\scriptstyle \delta^w} \\ F_*u & \xrightarrow{F_*g} & F_*w \end{array} \right\}$$

Outline

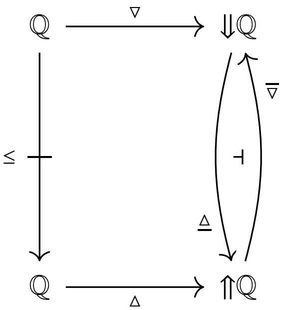
Dedekind's tight completion $\mathbb{Q} \hookrightarrow \mathbb{R}$

Loose completions

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\ \ \nabla\ \ } & \Downarrow\mathbb{Q} \\ {\scriptstyle \leq}\ \big\downarrow\!\!\!+ & & \\ \mathbb{Q} & \xrightarrow[\ \ \Delta\ \ ]{} & \Uparrow\mathbb{Q} \end{array}$$

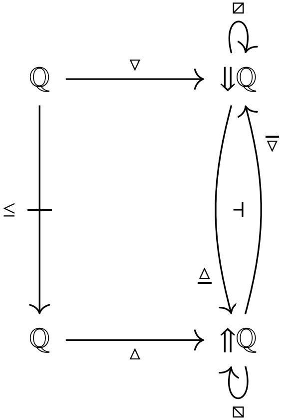
Dedekind's tight completion $\mathbb{Q} \hookrightarrow \mathbb{R}$

Galois connection



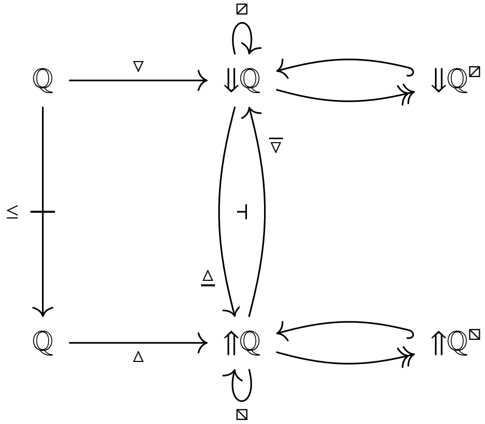
Dedekind's tight completion $\mathbb{Q} \hookrightarrow \mathbb{R}$

Closure operators



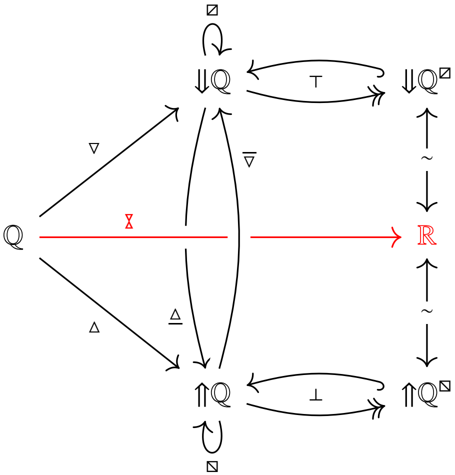
Dedekind's tight completion $\mathbb{Q} \hookrightarrow \mathbb{R}$

Closed elements

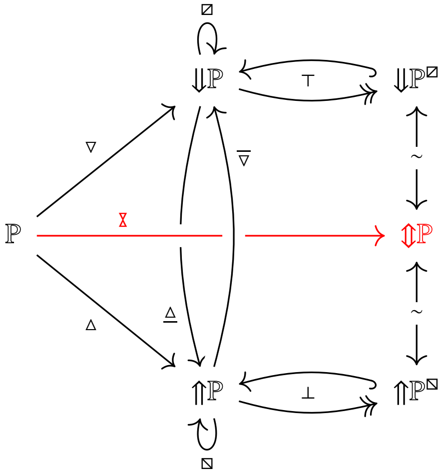


Dedekind's tight completion $\mathbb{Q} \hookrightarrow \mathbb{R}$

Cuts



MacNeille's tight completion $\mathbb{P} \xrightarrow{\text{⧖}} \Updownarrow\mathbb{P}$



MacNeille's tight completion $\mathbb{P} \xrightarrow{\between} \Updownarrow\mathbb{P}$

- $\Updownarrow\mathbb{P}$ has $\bigvee$ and $\bigwedge$
- $\Updownarrow\mathbb{P}$ is generated by $\bigvee$ and $\bigwedge$ of $\mathbb{P}$
- $\between$ preserves any $\bigvee$ and $\bigwedge$ that exist in $\mathbb{P}$

Tight bicompletion of a category $\mathbb{C} \xrightarrow{\between} \Updownarrow\mathbb{C}$

- $\Updownarrow\mathbb{C}$ has $\varinjlim$ and $\varprojlim$
- $\Updownarrow\mathbb{C}$ is generated by $\varinjlim$ and $\varprojlim$ of $\mathbb{C}$
- $\between$ preserves any $\varinjlim$s and $\varprojlim$s that exist in $\mathbb{C}$

Lambek's problem

"It is an open problem whether there exists a sup- and inf-complete category $\mathbb{A}''''$ with a sup- and inf-dense embedding $\mathbb{A} \to \mathbb{A}''''$ in analogy to the Dedekind completion of an ordered set."

Joachim Lambek

Completions of categories

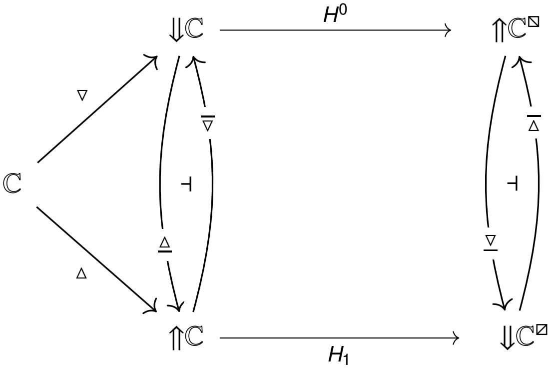
Lect. Notes in Math. Vol. 24, Springer 1966

Isbell's no-go theorem

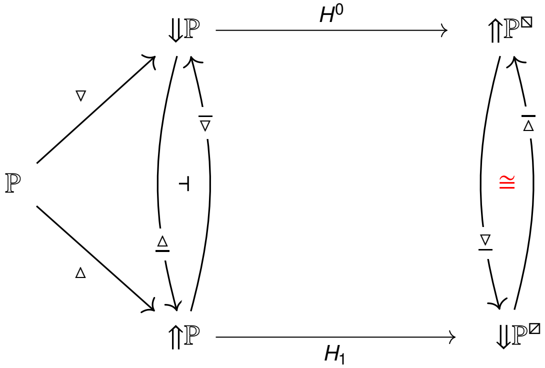
"No Lambek extension of the one-object category $\mathbb{Z}_4$ has finite limits."

John Isbell
Thm. 3.1. in *Small categories and completeness*.
Math. Syst. Theory 2(1), 1968

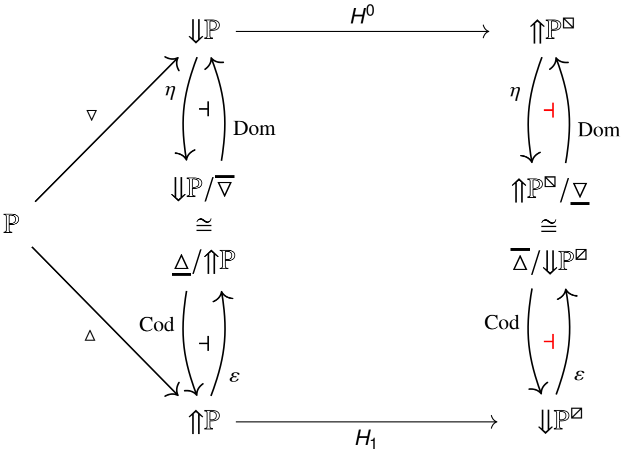
Loose completions ⇝ nucleus



In posets: nucleus ⇝ tight completion



Gaps vs intervals



Gaps vs intervals

$$\begin{array}{c} \Downarrow\mathbb{P}/\overline{\nabla} \\ \cong \\ \underline{\Delta}/\Uparrow\mathbb{P} \end{array} \; : \; \frac{\ell \subseteq \overline{\nabla} u}{\underline{\Delta}\ell \supseteq u} \qquad\qquad \frac{u \supseteq \underline{\Delta}\ell}{\overline{\nabla} u \subseteq \ell} \; : \; \begin{array}{c} \Uparrow\mathbb{P}^{\boxbslash}/\underline{\nabla} \\ \cong \\ \overline{\Delta}/\Downarrow\mathbb{P}^{\boxslash} \end{array}$$

Gaps vs intervals

$$\begin{array}{c} \Downarrow\mathbb{P}/\overline{\nabla} \\ \cong \\ \underline{\Delta}/\Uparrow\mathbb{P} \end{array} \; : \; \begin{array}{c} \ell \subseteq \overline{\nabla} u \\ \hline\hline \bigvee \ell \leq \bigwedge u \\ \hline\hline \underline{\Delta} \ell \supseteq u \end{array}$$

$$\begin{array}{c} u \supseteq \underline{\nabla} \ell \\ \hline\hline \bigwedge u \leq \bigvee \ell \\ \hline\hline \overline{\Delta} u \subseteq \ell \end{array} \; : \; \begin{array}{c} \Uparrow\mathbb{P}^{\boxbslash}/\underline{\nabla} \\ \cong \\ \overline{\Delta}/\Downarrow\mathbb{P}^{\boxslash} \end{array}$$

Gaps vs intervals

$$
\begin{array}{c}\Downarrow\mathbb{P}/\overline{\nabla}\\ \cong \\ \underline{\Delta}/\Uparrow\mathbb{P}\end{array} \; : \;
\dfrac{\dfrac{\ell \subseteq \overline{\nabla} u}{\bigvee \ell \leq \bigwedge u}}{\underline{\Delta}\ell \supseteq u}
\qquad
\dfrac{\dfrac{\ell = \overline{\nabla} u}{\bigvee \ell = \bigwedge u}}{\underline{\Delta}\ell = u}
\qquad
\dfrac{\dfrac{u \supseteq \underline{\Delta}\ell}{\bigwedge u \leq \bigvee \ell}}{\overline{\nabla} u \subseteq \ell}
\; : \;
\begin{array}{c}\Uparrow\mathbb{P}^{\boxbslash}/\underline{\nabla}\\ \cong \\ \overline{\Delta}/\Downarrow\mathbb{P}^{\boxslash}\end{array}
$$

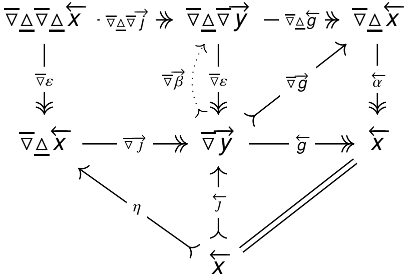
Proposition

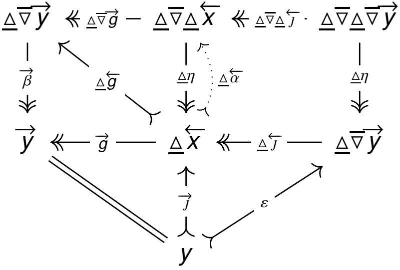
$$
\begin{array}{c} \Downarrow\mathbb{C}/\overline{\nabla} \\ \cong \\ \underline{\Delta}/\Uparrow\mathbb{C} \end{array}
\;:\;
\begin{array}{c} \overleftarrow{\jmath} : \overleftarrow{x} \to \overline{\nabla}\overrightarrow{y} \\ \updownarrow \\ \overrightarrow{\jmath} : \underline{\Delta}\overleftarrow{x} \leftarrow \overrightarrow{y} \end{array}
\qquad
\begin{array}{c} \overrightarrow{g} : \overrightarrow{\beta}_y \leftarrow \underline{\Delta}\overleftarrow{\alpha}_x \\ \updownarrow \\ \overleftarrow{g} : \overline{\nabla}\overrightarrow{\beta}_y \to \overleftarrow{\alpha}_x \end{array}
\;:\;
\begin{array}{c} \Uparrow\mathbb{C}^{\boxbslash}/\underline{\nabla} \\ \cong \\ \overline{\Delta}/\Downarrow\mathbb{C}^{\boxslash} \end{array}
$$

then

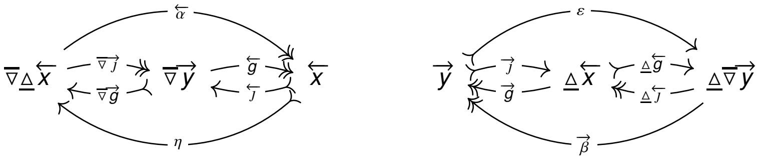
$$
\left\{\begin{array}{l} \overleftarrow{g} \circ \overleftarrow{\jmath} = \mathrm{id}_x \;\wedge \\ \overrightarrow{\beta}_y \circ \underline{\Delta}\left(\overleftarrow{\jmath} \circ \overleftarrow{g}\right) = \overrightarrow{\beta}_y \end{array}\right\}
\iff
\left\{\begin{array}{l} \overrightarrow{g} \circ \overrightarrow{\jmath} = \mathrm{id}_y \;\wedge \\ \overleftarrow{\alpha}_x \circ \overline{\nabla}\left(\overrightarrow{\jmath} \circ \overrightarrow{g}\right) = \overleftarrow{\alpha}_x \end{array}\right\}
$$

Proof





Corollary



Category of cuts

$$|\Updownarrow\mathbb{C}| \quad = \quad \coprod_{\substack{\overleftarrow{\alpha}\in\Downarrow\mathbb{C}^{\boxslash} \\ \overrightarrow{\alpha}\in\Uparrow\mathbb{C}^{\boxbslash}}} \left\{ \overleftarrow{\alpha} \underset{\overleftarrow{\jmath}}{\overset{\overleftarrow{g}}{\leftleftarrows}} \underline{\nabla}\overrightarrow{\alpha} \quad \wedge \quad \overline{\Delta}\overleftarrow{\alpha} \underset{\overrightarrow{\jmath}}{\overset{\overrightarrow{g}}{\rightrightarrows}} \overrightarrow{\alpha} \right\}$$

$$\Updownarrow\mathbb{C}(\alpha,\beta) \quad = \quad \coprod_{\substack{\overleftarrow{f}\in\Downarrow\mathbb{C}^{\boxslash}(\overleftarrow{\alpha},\overleftarrow{\beta}) \\ \overrightarrow{f}\in\Uparrow\mathbb{C}^{\boxbslash}(\overrightarrow{\alpha},\overrightarrow{\beta})}} \left\{ \begin{array}{ccccc} \overleftarrow{x} & \rightarrowtail \overleftarrow{\jmath}_{\alpha} \rightarrow & \overline{\nabla}\overrightarrow{x} & - \overleftarrow{g}_{\alpha} \twoheadrightarrow & \overleftarrow{x} \\ \Big\downarrow \overleftarrow{f} & & \Big\downarrow \overline{\nabla}\overrightarrow{f} & & \Big\downarrow \overleftarrow{f} \\ \overleftarrow{y} & \rightarrowtail \overleftarrow{\jmath}_{\beta} \rightarrow & \overline{\nabla}\overrightarrow{y} & - \overleftarrow{g}_{\beta} \twoheadrightarrow & \overleftarrow{y} \end{array} \right\}$$

Category of cuts

$$|\Updownarrow\mathbb{C}| \quad = \quad \coprod_{\substack{\overleftarrow{\alpha}\in\Downarrow\mathbb{C}^{\boxslash} \\ \overrightarrow{\alpha}\in\Uparrow\mathbb{C}^{\boxbslash}}} \left\{ \overleftarrow{\alpha} \underset{\overleftarrow{\jmath}}{\overset{\overleftarrow{g}}{\twoheadleftarrow}} \underline{\nabla}\overrightarrow{\alpha} \quad \wedge \quad \overline{\Delta}\overleftarrow{\alpha} \underset{\overrightarrow{\jmath}}{\overset{\overrightarrow{g}}{\twoheadrightarrow}} \overrightarrow{\alpha} \right\}$$

$$\Updownarrow\mathbb{C}(\alpha,\beta) \quad = \quad \coprod_{\substack{\overleftarrow{f}\in\Downarrow\mathbb{C}^{\boxslash}(\overleftarrow{\alpha},\overleftarrow{\beta}) \\ \overrightarrow{f}\in\Uparrow\mathbb{C}^{\boxbslash}(\overrightarrow{\alpha},\overrightarrow{\beta})}} \left\{ \begin{array}{ccccc} \overrightarrow{x} & \twoheadleftarrow \overrightarrow{g}_{\alpha} - & \underline{\Delta}\overleftarrow{x} & \leftarrow \overrightarrow{\jmath}_{\alpha} \prec & \overrightarrow{x} \\ \uparrow & & \uparrow & & \uparrow \\ \overrightarrow{f} & & \underline{\Delta}\overleftarrow{f} & & \overrightarrow{f} \\ | & & | & & | \\ \overrightarrow{y} & \twoheadleftarrow \overrightarrow{g}_{\beta} - & \underline{\Delta}\overleftarrow{y} & \leftarrow \overrightarrow{\jmath}_{\beta} \prec & \overrightarrow{y} \end{array} \right\}$$

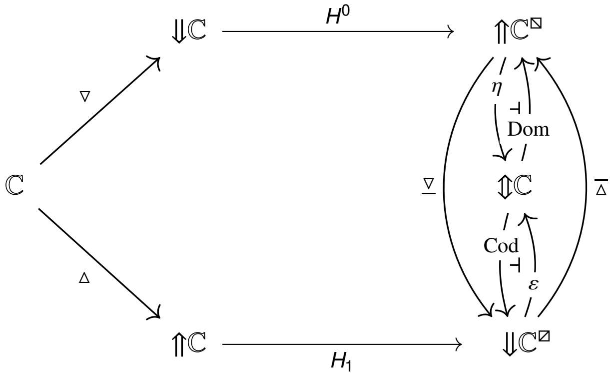
Lattice of cuts

$$\Updownarrow\mathbb{P} \quad = \quad \coprod_{\substack{\ell \in \Downarrow\mathbb{P} \\ u \in \Uparrow\mathbb{P}}} \left\{ \underline{\Delta}\ell = u \quad \wedge \quad \ell = \overline{\nabla} u \right\}$$

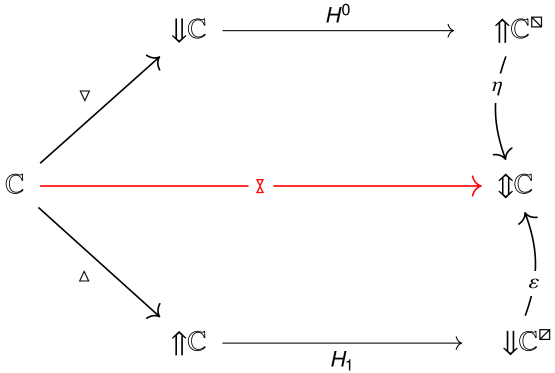
$$\ell_0 \subseteq \ell_1 \quad \iff \quad \langle \ell_0, u_0 \rangle \leq \langle \ell_1, u_1 \rangle \quad \iff \quad u_1 \subseteq u_0$$

Lemma

The nucleus factors through cuts



Tight completion



Theorem

$\mathbb{C} \xrightarrow{\Join} \Updownarrow\mathbb{C}$ is the tight bicompletion

- $\Updownarrow\mathbb{C}$ has $\overrightarrow{\lim}$ and $\overleftarrow{\lim}$
- $\Updownarrow\mathbb{C}$ is $\overrightarrow{\lim}$-generated and $\overleftarrow{\lim}$-generated by $\mathbb{C}$
- $\Join$ preserves any $\overleftarrow{\lim}$ and $\overrightarrow{\lim}$ that exist in $\mathbb{C}$

Loose limits and colimits

$$\mathbb{C} \;\underset{\nabla}{\overset{\underrightarrow{\lim}}{\leftrightarrows}}\; \Downarrow\mathbb{C} \qquad (\bot)$$

$$\mathbb{C}(\underrightarrow{\lim}\,\overleftarrow{\alpha}, y) \cong \Downarrow\mathbb{C}(\overleftarrow{\alpha}, \nabla y)$$

$$\mathbb{C} \;\underset{\vartriangle}{\overset{\underleftarrow{\lim}}{\leftrightarrows}}\; \Uparrow\mathbb{C} \qquad (\top)$$

$$\mathbb{C}(x, \underleftarrow{\lim}\,\overrightarrow{\beta}) \cong \Uparrow\mathbb{C}(\vartriangle x, \overrightarrow{\beta})$$

Tight limits and colimits

$$\mathbb{C} \;\underset{\text{⧖}}{\overset{\overrightarrow{\lim}}{\underset{\bot}{\leftrightarrows}}}\; \Updownarrow\mathbb{C} \qquad\qquad \mathbb{C} \;\underset{\text{⧖}}{\overset{\overleftarrow{\lim}}{\underset{\top}{\leftrightarrows}}}\; \Updownarrow\mathbb{C}$$

$$\mathbb{C}\big(\overrightarrow{\lim_{\varphi}}\, \overleftarrow{\alpha}, y\big) \cong \Updownarrow\mathbb{C}\big(\varphi_{\overleftarrow{\alpha}}, \mathrm{id}_{\text{⧖}y}\big) \qquad\qquad \mathbb{C}\big(x, \overleftarrow{\lim_{\psi}}\, \overrightarrow{\beta}\big) \cong \Updownarrow\mathbb{C}\big(\mathrm{id}_{\text{⧖}x}, \psi_{\overrightarrow{\beta}}\big)$$

Outline

Background: Decompositions

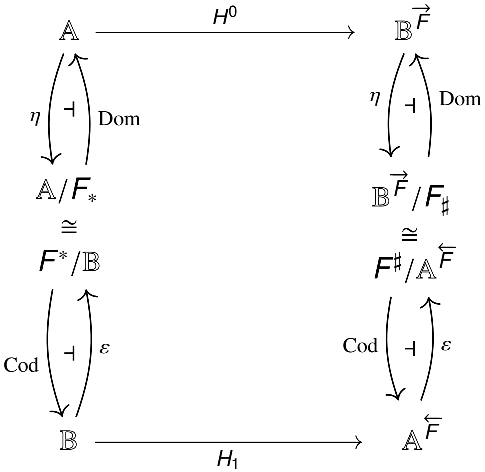
New problem: Concept mining

Approach: Nuclear adjunctions

Old problem: Tight completions

Solution: Categorical cuts and concepts

Concept cuts



Concept cuts

$$\begin{array}{ccc} \mathbb{A}/F_* & & \overleftarrow{j} : x \to F_* y \\ \cong & : & \updownarrow \\ F^*/\mathbb{B} & & \overrightarrow{j} : F^* x \leftarrow y \end{array} \qquad \begin{array}{ccc} \overrightarrow{h} : \overrightarrow{\beta}_y \leftarrow F^* \overleftarrow{\alpha}_x & & \mathbb{B}^{\overrightarrow{F}}/F_\sharp \\ \updownarrow & : & \cong \\ \overleftarrow{h} : F_* \overrightarrow{\beta}_y \to \overleftarrow{\alpha}_x & & F^\sharp/\mathbb{A}^{\overleftarrow{F}} \end{array}$$

with

$$\left\{ \begin{array}{l} \overleftarrow{h} \circ \overleftarrow{\jmath} = \mathrm{id}_x \ \wedge \\ \overrightarrow{\beta}_y \circ F^* \left(\overleftarrow{\jmath} \circ \overleftarrow{h} \right) = \overrightarrow{\beta}_y \end{array} \right\} \iff \left\{ \begin{array}{l} \overrightarrow{h} \circ \overrightarrow{\jmath} = \mathrm{id}_y \ \wedge \\ \overleftarrow{\alpha}_x \circ F_* \left(\overrightarrow{\jmath} \circ \overrightarrow{h} \right) = \overleftarrow{\alpha}_x \end{array} \right\}$$

Concept category

$$|\Updownarrow\Phi| \;=\; \coprod_{\substack{\overleftarrow{\alpha}\in\Downarrow\mathcal{B}^{\overleftarrow{\Phi}}\\ \overrightarrow{\alpha}\in\Uparrow\mathcal{A}^{\overrightarrow{\Phi}}}} \left\{ \overleftarrow{\alpha} \underset{\overleftarrow{\jmath}}{\overset{\overleftarrow{g}}{\leftrightarrows}} \Phi_\sharp\overrightarrow{\alpha} \quad\wedge\quad \Phi^\sharp\overleftarrow{\alpha} \underset{\overrightarrow{\jmath}}{\overset{\overrightarrow{g}}{\rightleftarrows}} \overrightarrow{\alpha} \right\}$$

$$\Updownarrow\Phi(\alpha,\beta) \;=\; \coprod_{\substack{\overleftarrow{f}\in\Downarrow\mathcal{B}^{\overleftarrow{\Phi}}(\overleftarrow{\alpha},\overleftarrow{\beta})\\ \overrightarrow{f}\in\Uparrow\mathcal{A}^{\overrightarrow{\Phi}}(\overrightarrow{\alpha},\overrightarrow{\beta})}} \left\{ \begin{array}{ccccc} \overleftarrow{x} & \stackrel{\overleftarrow{\jmath}_\alpha}{\rightarrowtail} & \Phi_*\overrightarrow{x} & \stackrel{\overleftarrow{g}_\alpha}{\twoheadrightarrow} & \overleftarrow{x} \\ \downarrow{\scriptstyle \overleftarrow{f}} & & \downarrow{\scriptstyle \Phi_*\overrightarrow{f}} & & \downarrow{\scriptstyle \overleftarrow{f}} \\ \overleftarrow{y} & \stackrel{\overleftarrow{\jmath}_\beta}{\rightarrowtail} & \Phi_*\overrightarrow{y} & \stackrel{\overleftarrow{g}_\beta}{\twoheadrightarrow} & \overleftarrow{y} \end{array} \right\}$$

Concept category

$$|\Updownarrow\Phi| \quad = \coprod_{\substack{\overleftarrow{\alpha}\in\Downarrow\mathcal{B}^{\overleftarrow{\Phi}}\\ \overrightarrow{\alpha}\in\Uparrow\mathcal{A}^{\overrightarrow{\Phi}}}} \left\{ \overleftarrow{\alpha} \underset{\overleftarrow{\jmath}}{\overset{\overleftarrow{g}}{\twoheadleftarrow}} \Phi_\sharp\overrightarrow{\alpha} \quad \wedge \quad \Phi^\sharp\overleftarrow{\alpha} \underset{\overleftarrow{\jmath}}{\overset{\overrightarrow{g}}{\twoheadrightarrow}} \overrightarrow{\alpha} \right\}$$

$$\Updownarrow\Phi(\alpha,\beta) \quad = \coprod_{\substack{\overleftarrow{f}\in\Downarrow\mathcal{B}^{\overleftarrow{\Phi}}(\overleftarrow{\alpha},\overleftarrow{\beta})\\ \overrightarrow{f}\in\Uparrow\mathcal{A}^{\overrightarrow{\Phi}}(\overrightarrow{\alpha},\overrightarrow{\beta})}} \left\{ \begin{array}{ccccc} \overrightarrow{x} & \twoheadleftarrow \overrightarrow{g}_\alpha - & \Phi^*\overleftarrow{x} & \leftarrow \overrightarrow{\jmath}_\alpha \prec & \overrightarrow{x} \\ \uparrow & & \uparrow & & \uparrow \\ \overrightarrow{f} & & \Phi^*\overleftarrow{f} & & \overrightarrow{f} \\ | & & | & & | \\ \overrightarrow{y} & \twoheadleftarrow \overrightarrow{g}_\beta - & \Phi^*\overleftarrow{y} & \leftarrow \overrightarrow{\jmath}_\beta \prec & \overrightarrow{y} \end{array} \right\}$$

References

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- DP and D. Hughes, *Tight limits and completions*. https://arxiv.org/abs/2204.09285