Game Spectrum of a Monad 

Alyssa Renata (She/Her)
(Imperial College London)

Pacific Category Theory (PCT) Seminar 28 August 2026

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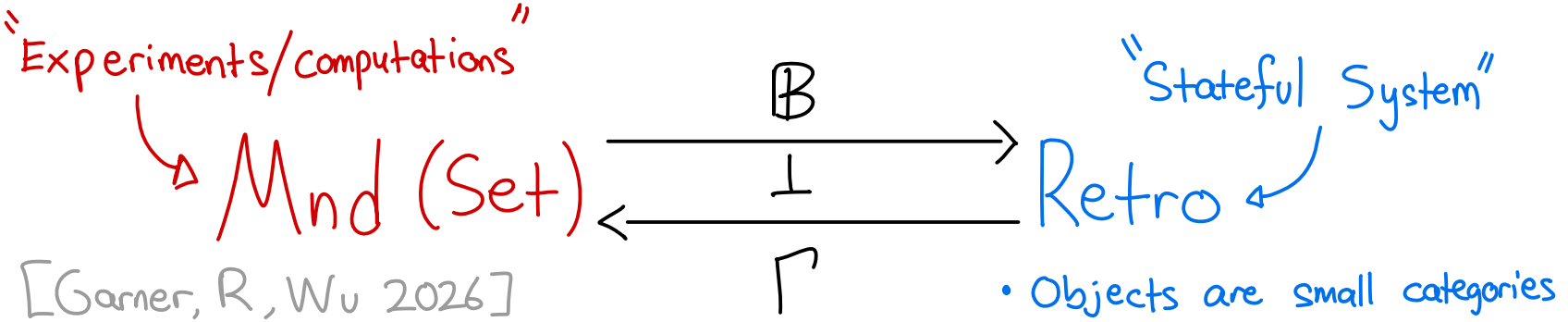

[Garner, R, Wu 2026]

- Objects are small categories
- morphisms are retrofunctors (AKA cofunctors)

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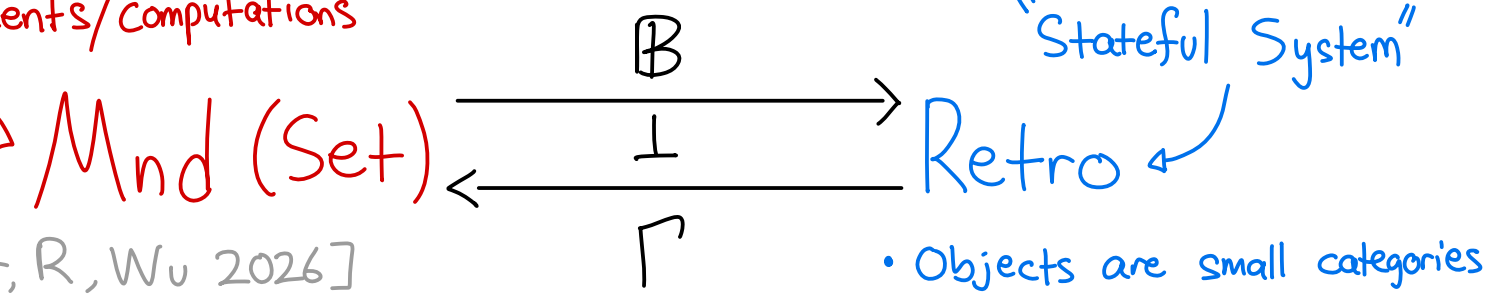
"Experiments/computations"



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"Measurements/functions"

"State space"

- Analogy $\mathbf{CRing} \simeq \mathbf{AffScheme}^{op}$

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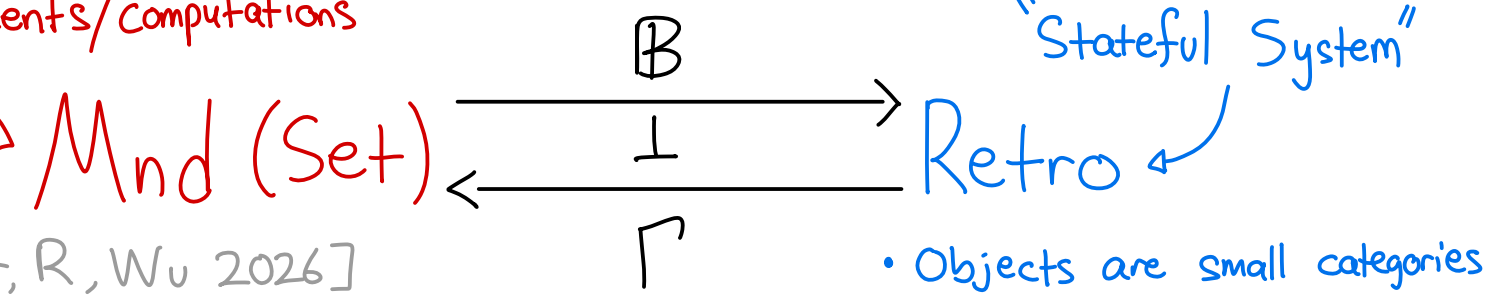

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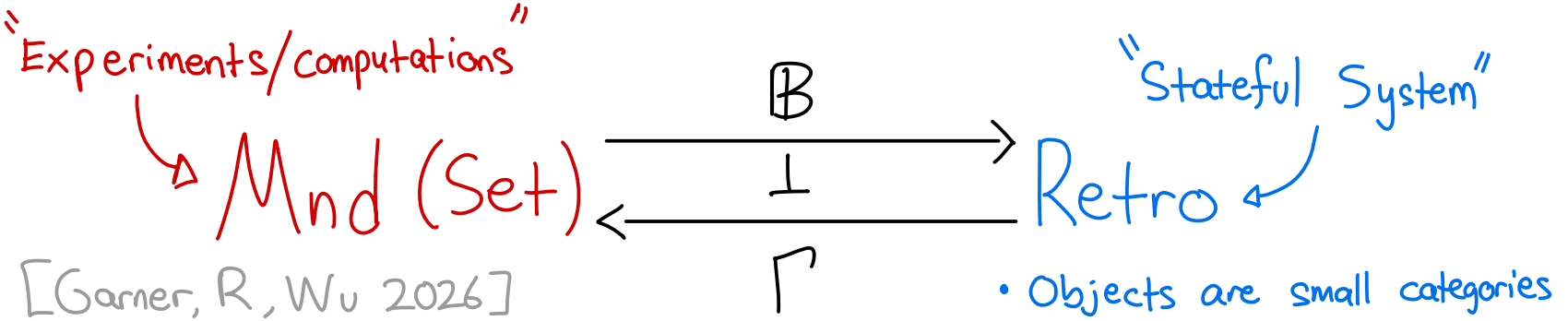

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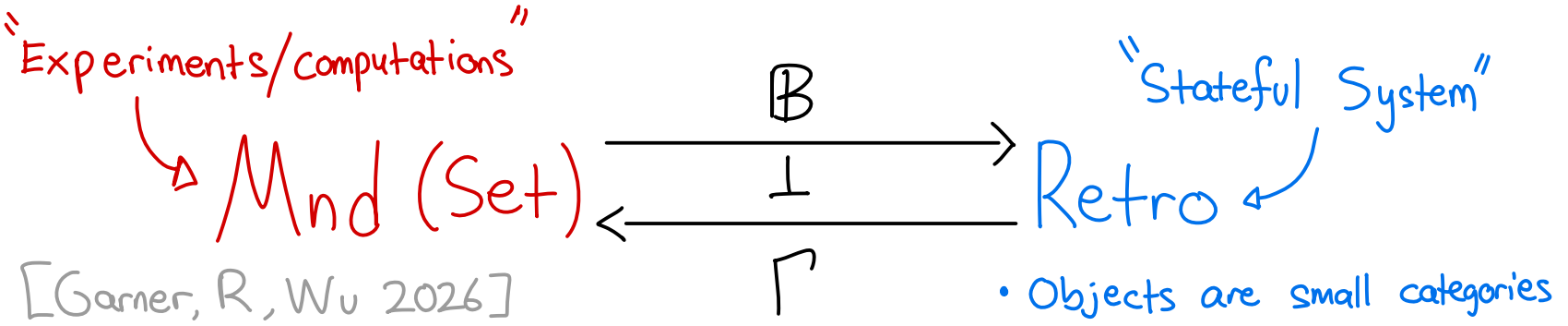

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"Experiments/computations"

Mnd (Set)

[Garner, R, Wu 2026]

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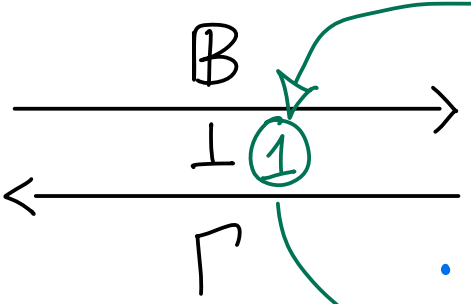

"Stateful System"

Retro

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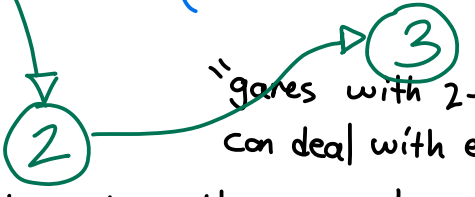
"state space"



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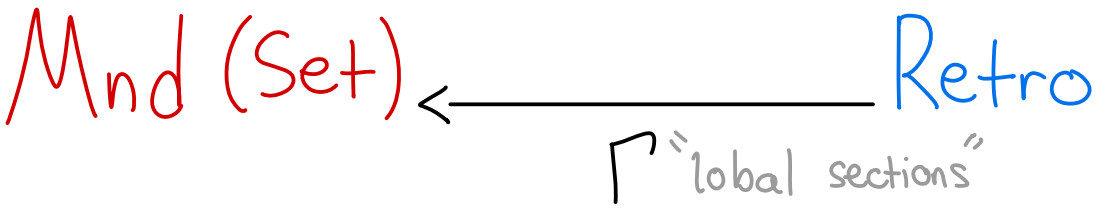
Plan for the Talk

① The Behaviour Category of a Monad



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$\mathbf{Mnd}(\mathbf{Set}) \longleftarrow \mathbf{Retro}$

$\ulcorner$ "lobal sections"

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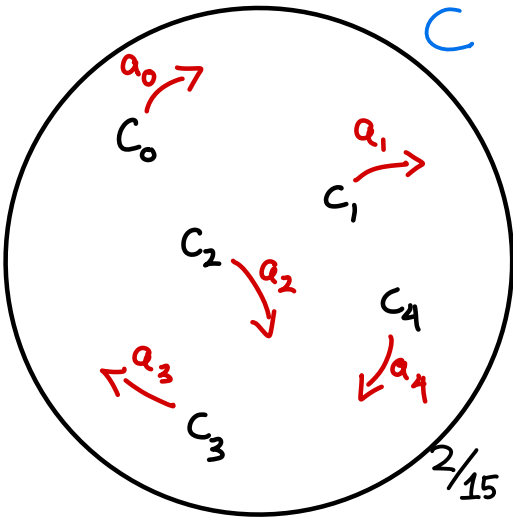
$$\ulcorner C(A) = \left\{ C_0 \xrightarrow{s} C_1 \times A \;\middle|\; \begin{array}{ccc} C_0 & \xrightarrow{s} & C_1 \times A \\ \| & & \downarrow \pi \\ C_0 & \xleftarrow[src]{} & C_1 \end{array} \right\}$$



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Γ "global sections"

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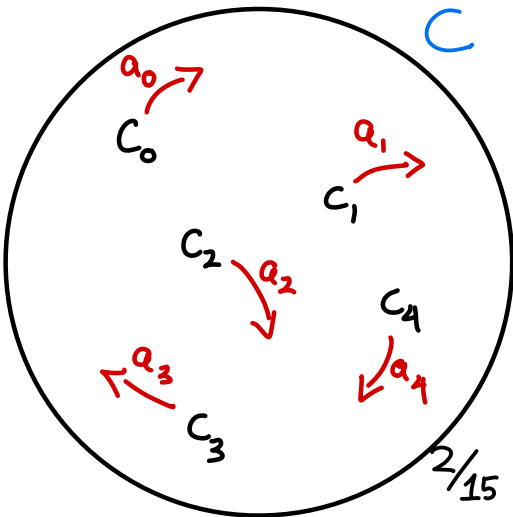
Special case Let $W \in \mathbf{Set}$.

$$\mathbb{C}\mathrm{odisc}(W) = \quad W \xleftarrow{\sigma} W \times W \xrightarrow{\tau} W$$



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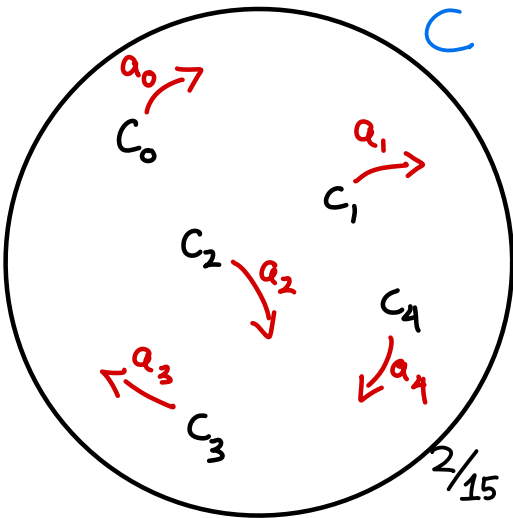
$$\mathbf{Codisc}(W) = \quad W \times W \;\overset{\sigma}{\swarrow}\; W \qquad W \times W \;\overset{\tau}{\searrow}\; W$$

$$\Gamma\,\mathbf{Codisc}(W)(A) = \left\{ W \xrightarrow{s} W \times W \times A \;\middle|\; \begin{array}{ccc} W & \longrightarrow & W \times W \times A \\ & \searrow & \downarrow \pi_1 \\ & & W \end{array} \right\}$$



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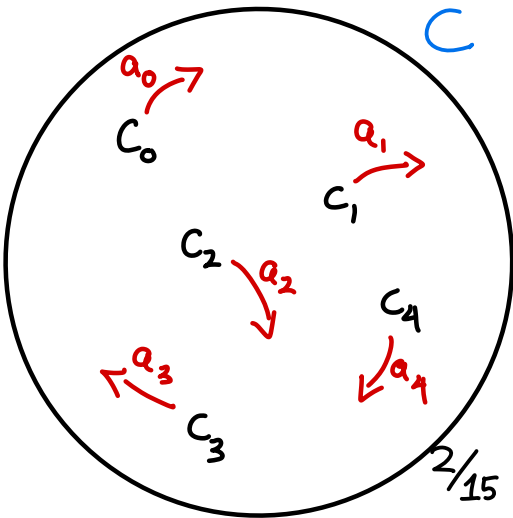
$$\cong (W \times A)^W$$

"State Monad"



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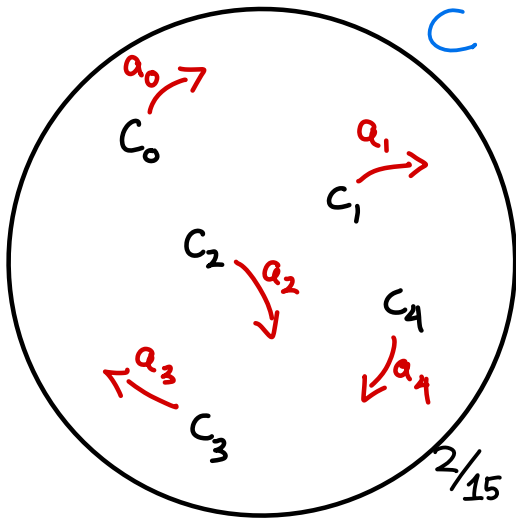

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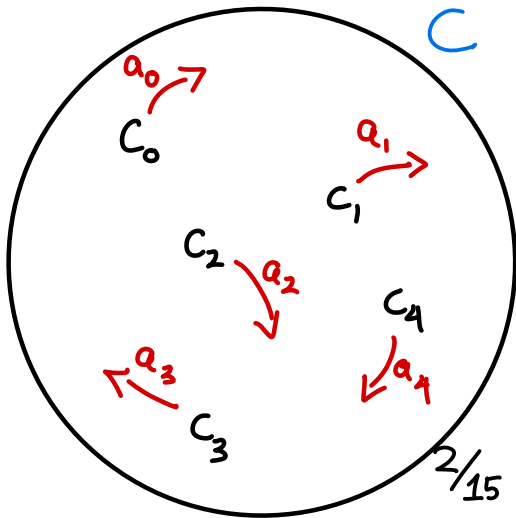

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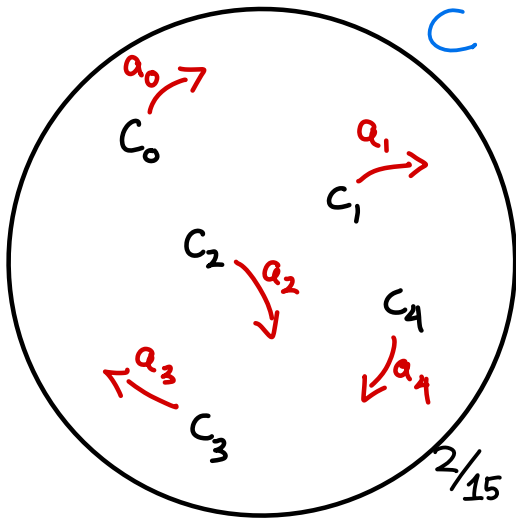

$$\mathrm{Mnd}(T, \Gamma C) \cong \mathrm{Retro}(\mathbb{B}T, C)$$

Bad example 1

Suppose $z \in T\emptyset$.

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$\rho(b)$ followed by $W \times \mathrm{flip}$ gives $W \times 2$

so if $\exists \omega \in W$ then $0 = 1$. i.e. $W = \emptyset$.

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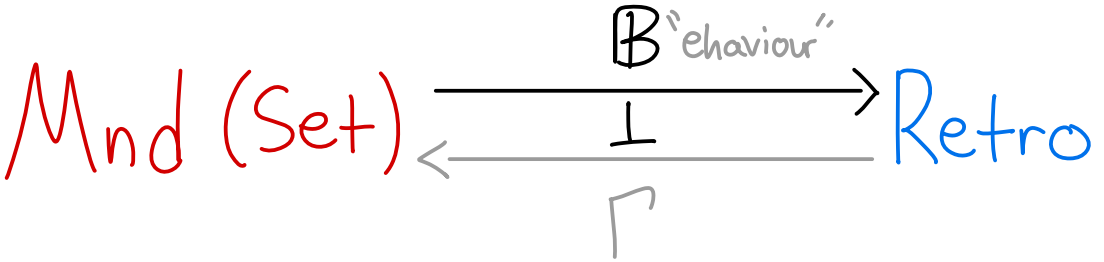
defn [Power-Shkaravska 2004] A T-comodel is a pair $(W, T \xrightarrow{\rho} (W \times -)^W)$.

(morphisms are $W \to W'$ commuting with the ρ)

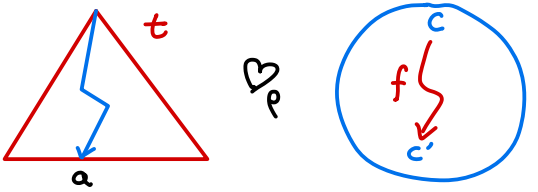

B "behaviour"
Mnd (Set)
⊥
Retro
⌜

$\mathbb{B}$ "behaviour"

$\mathbf{Mnd}(\mathbf{Set}) \rightleftarrows \mathbf{Retro}$ $\quad \dashv$

Given $T \xrightarrow{\rho} \Gamma C$,

From T's perspective, $c \in C_0$ tells it how to <u>run</u> a T-term $t \in TA$



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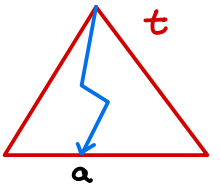
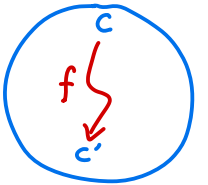
so $\mathbb{B}_0 T = \{ \beta : T \to \mathrm{Id} \mid$ 

$\} \heartsuit_\rho$ 

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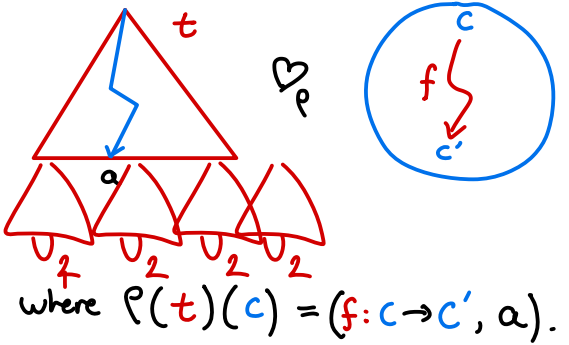
$1 \xrightarrow{t} A \xrightarrow{u} B$ Kleisli composition

$1 \xrightarrow{t} A \xrightarrow{!} 1 \xrightarrow{u(\beta(t))} B$



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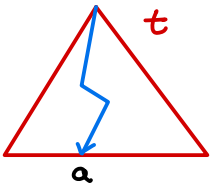
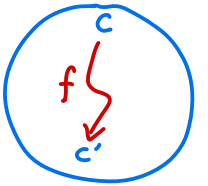


$\heartsuit_\rho$



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From c's perspective, t and t' are indistinguishable as long as they induce the same f.

so $\mathbb{B}_1 T = \coprod_{\beta \in \mathbb{B}_0 T} T1 / \sim_\beta$

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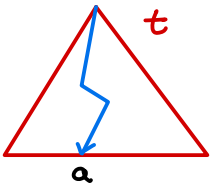
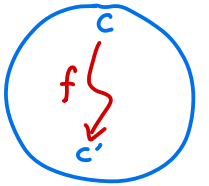


$\heartsuit_\rho$



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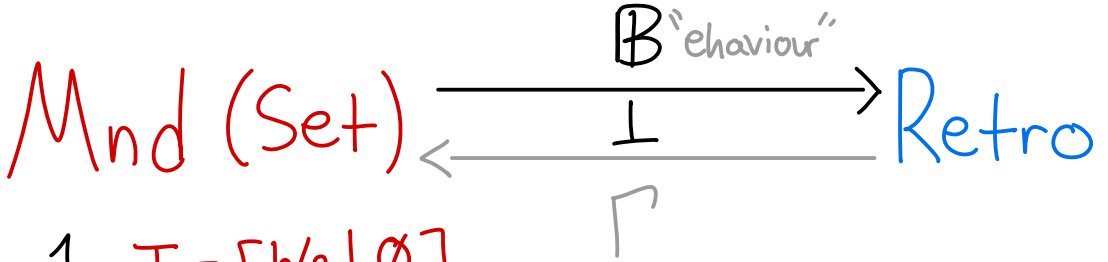
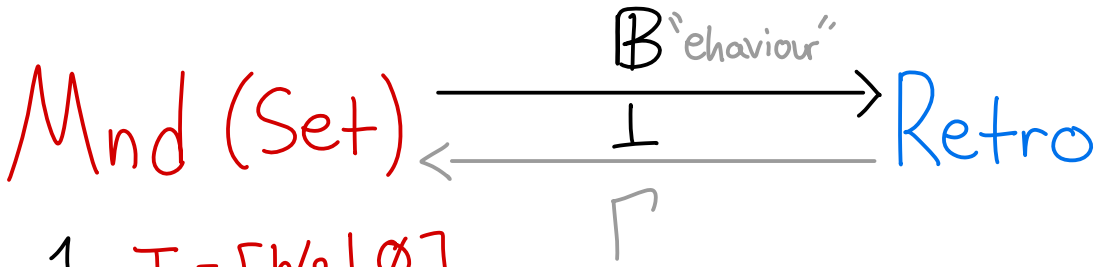


Example 1 $T = [b/2 \mid \emptyset]$



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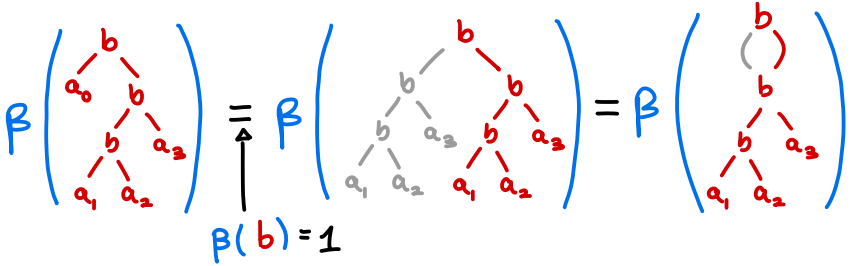
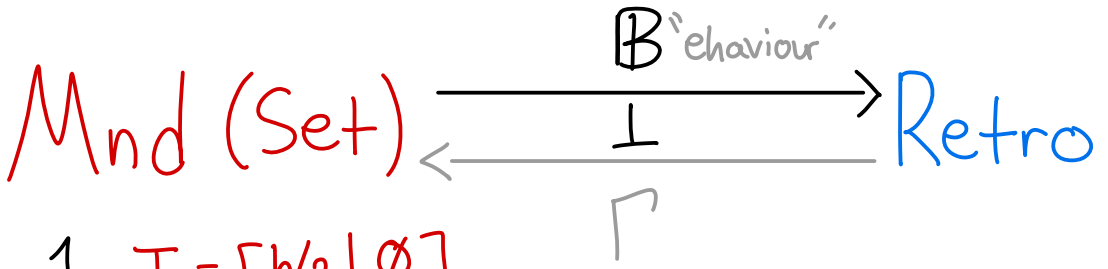
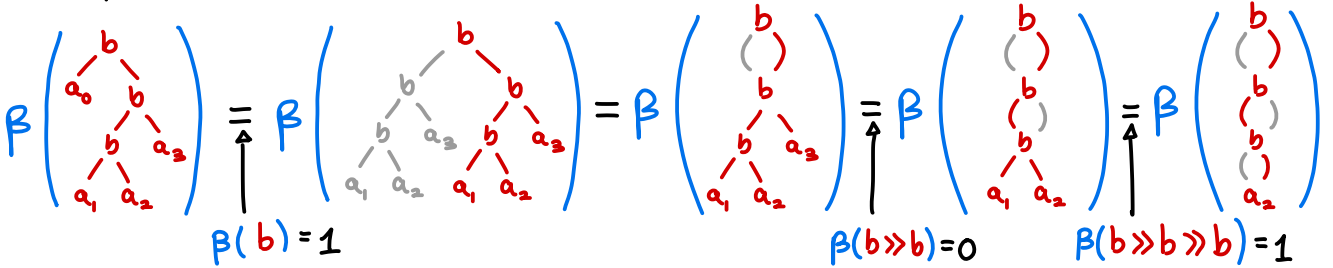
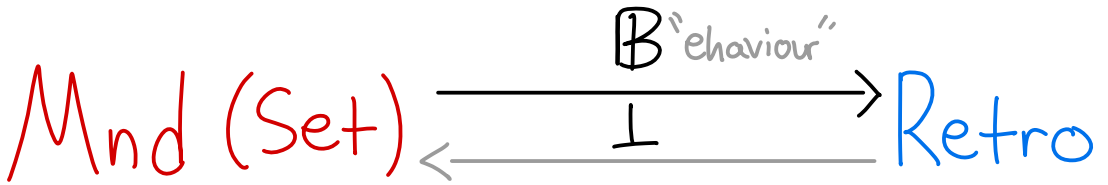


$\beta(b) = 1$



Example 1 $T = [b/2 \mid \emptyset]$





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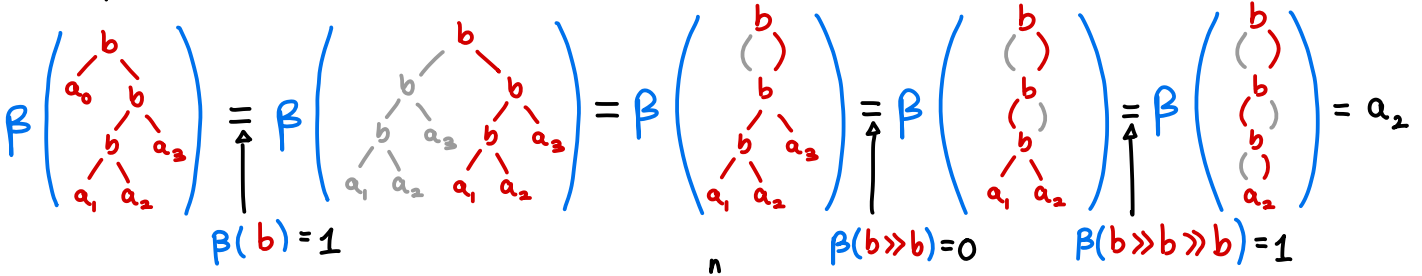
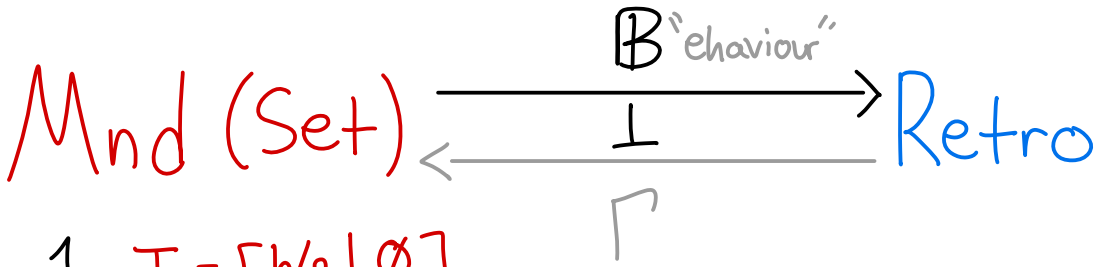


So β determined by $\forall n \in \omega.\ \beta(\overbrace{b \gg \cdots \gg b}^{n})$ i.e. $\mathcal{B}_0 T \cong 2^\omega$.



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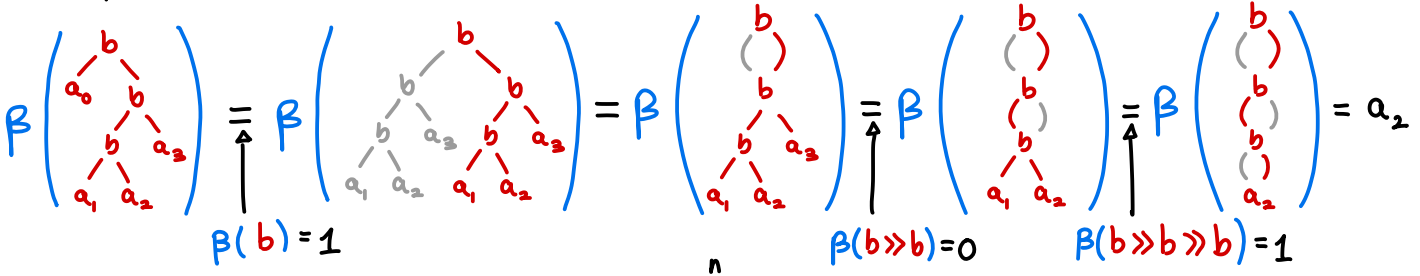

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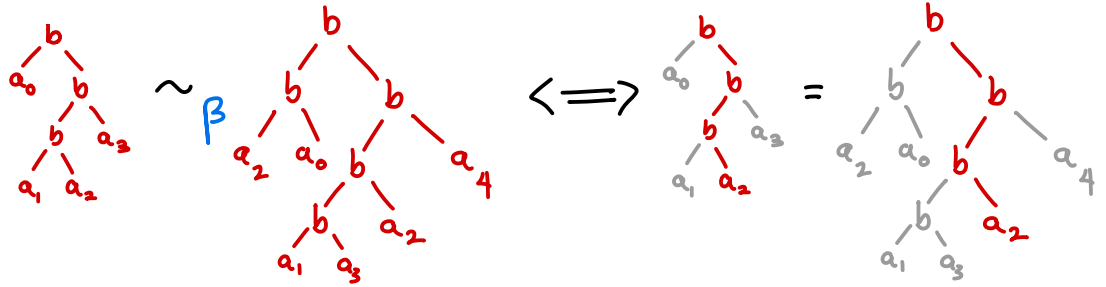

$\mathbb{B}$ "behaviour"

$\mathbf{Mnd}(\mathbf{Set}) \rightleftarrows \mathbf{Retro}$ (⊣)

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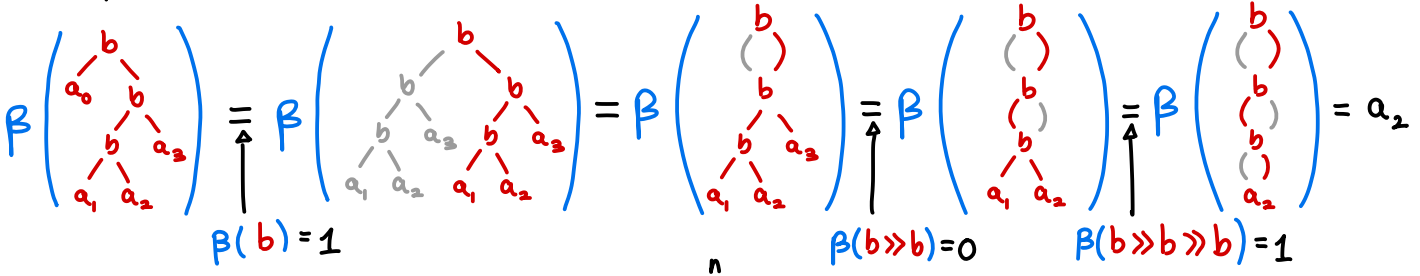

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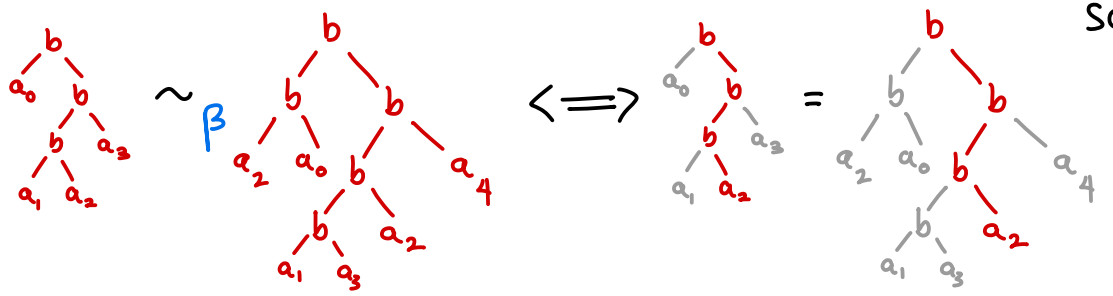
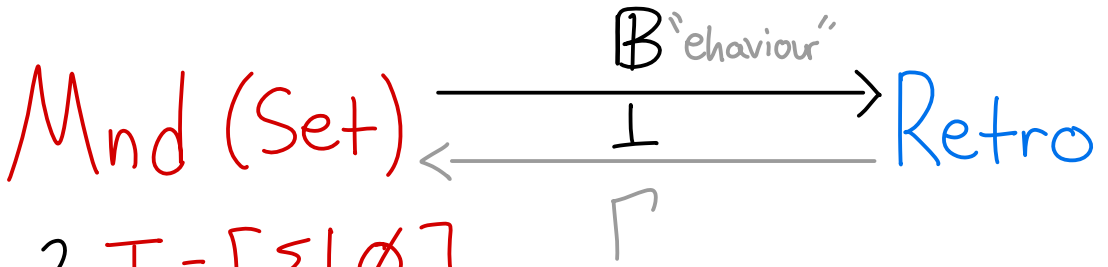


So $\mathbb{B}_1 T = \coprod_{\beta \in \mathbb{B}_0 T} T1/{\sim_\beta}$

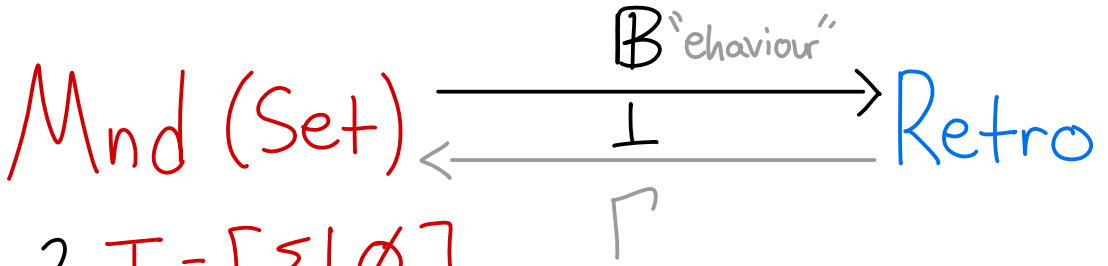
$\cong \coprod_{\beta} \mathbb{N} = \mathbb{B}_0 T \times \mathbb{N}$

ω

$n_\beta : \beta \to \mathrm{tail}^n \beta$



Example 2 $T = [\Sigma \mid \emptyset]$



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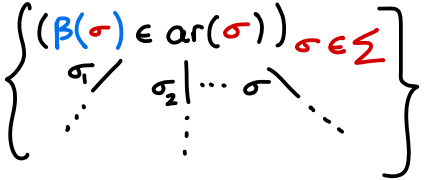

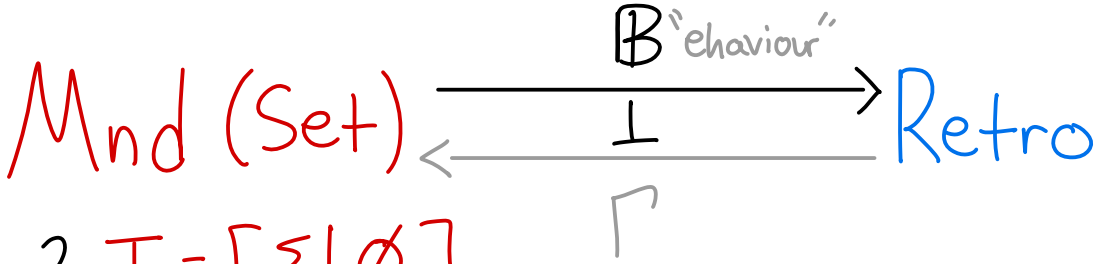

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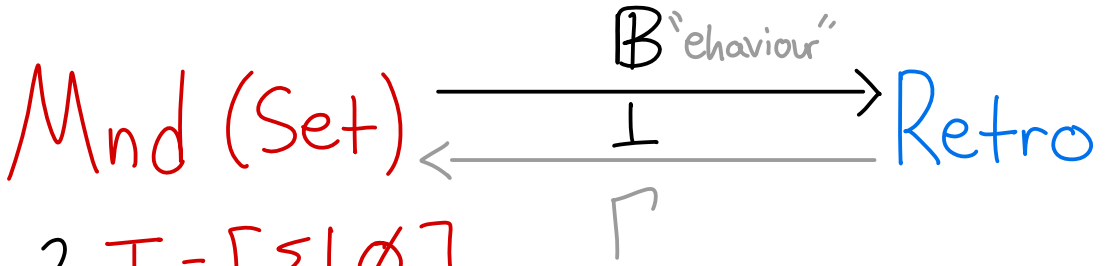

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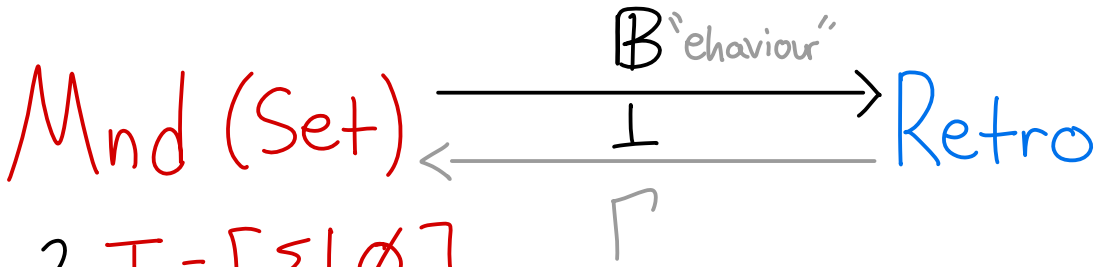
Example 3

- $T = (W \times -)^W$

$$\Longrightarrow \quad \mathbb{B} T = \text{codisc}(W)$$



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- $T = (W \times -)^W \implies \mathbb{B}T = \mathrm{codisc}(W)$
- $T = \mathcal{D}_\omega^B$ for Boolean Algebra $B \implies \mathbb{B}T = \mathrm{disc}(\mathrm{Spec}(B))$

$\mathbb{B}$ "behaviour"

$\mathbf{Mnd}(\mathbf{Set}) \rightleftarrows \mathbf{Retro}$ ($\dashv$)

We also have a topological/localic version generalizing Stone Duality!

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2 Games from Algebraic Signatures

Our intuition for $\mathbb{B}T$ is very interactive... Can we formalize this?

Example Let A be a set, $T = [b/2 \mid \emptyset]$.

Our intuition for $\mathbb{BT}$ is very interactive... Can we formalize this?

Example Let A be a set, $T = [b/2 \mid \emptyset]$.

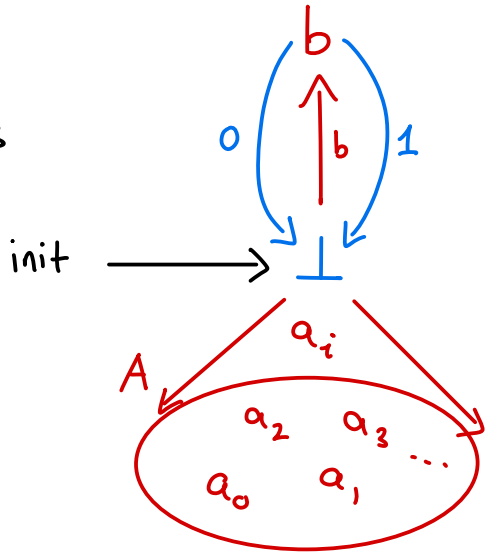
Consider the following alternating 2-player game $\mathcal{G}_A T$:

nodes are positions

edges are moves

= Program

= Environment



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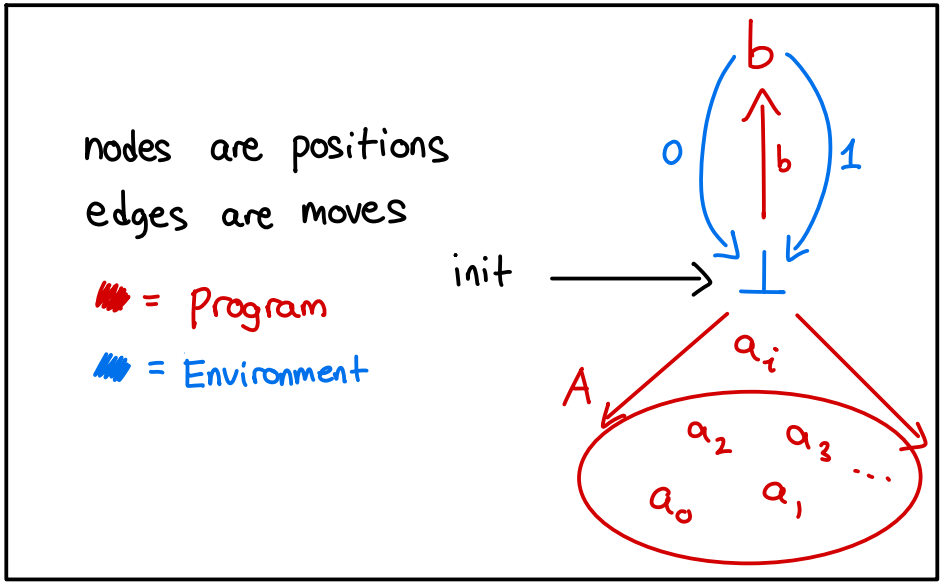

winning strategies for program:

$\xrightarrow{b}$

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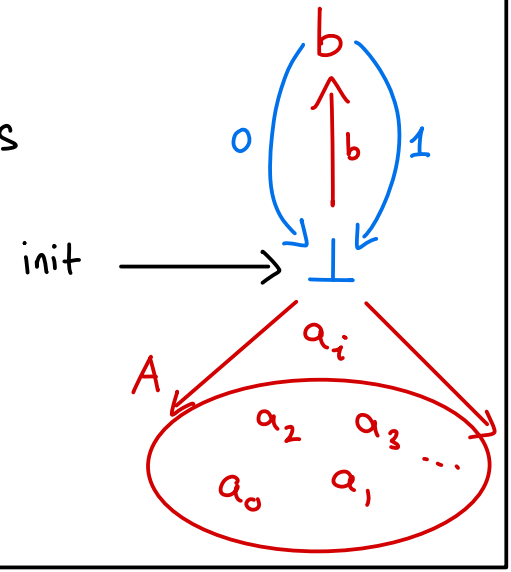

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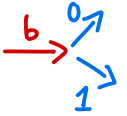

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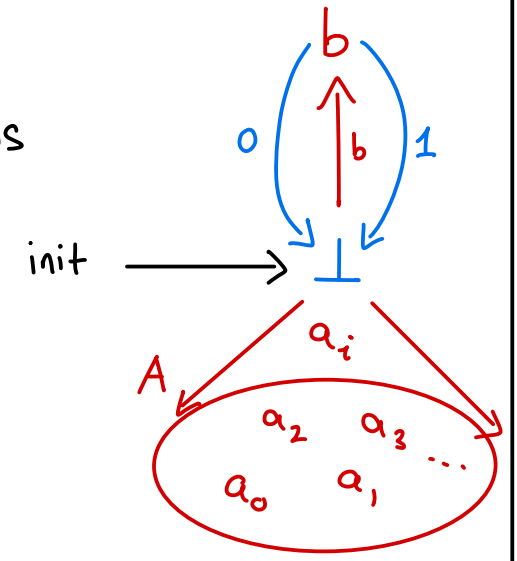

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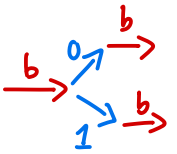

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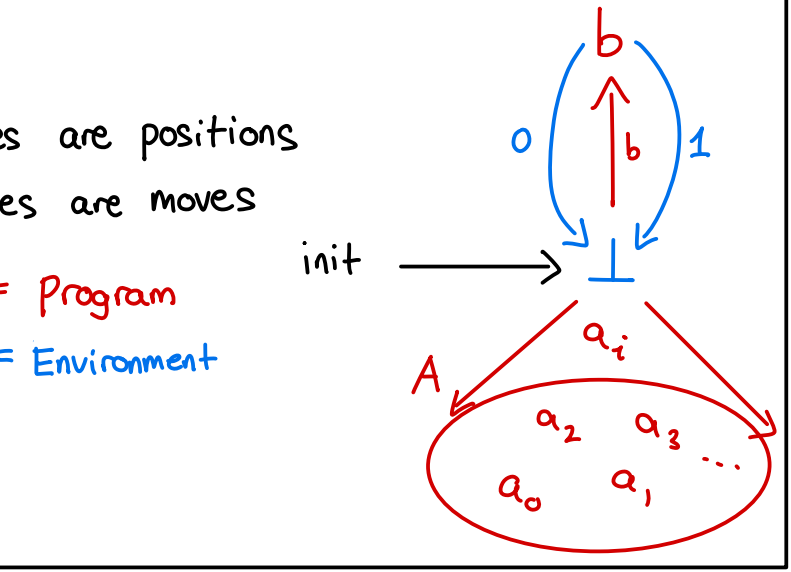

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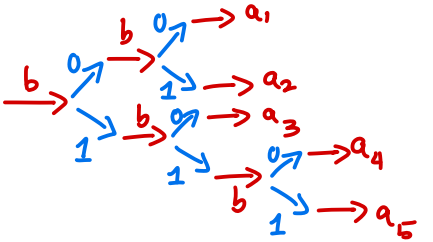

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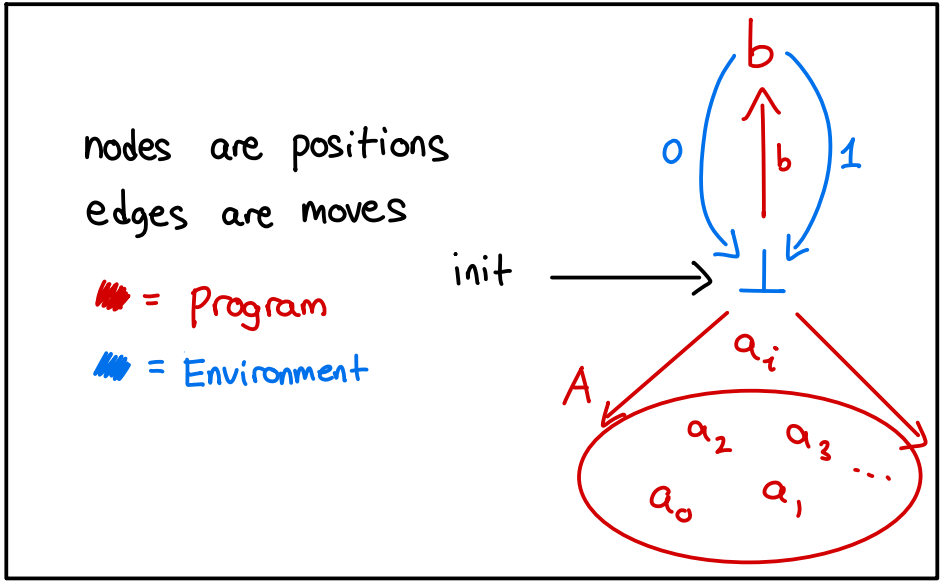

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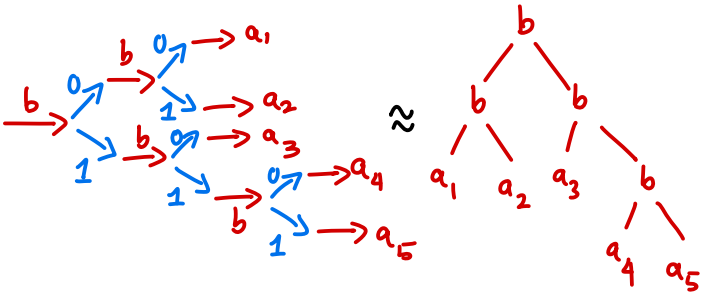

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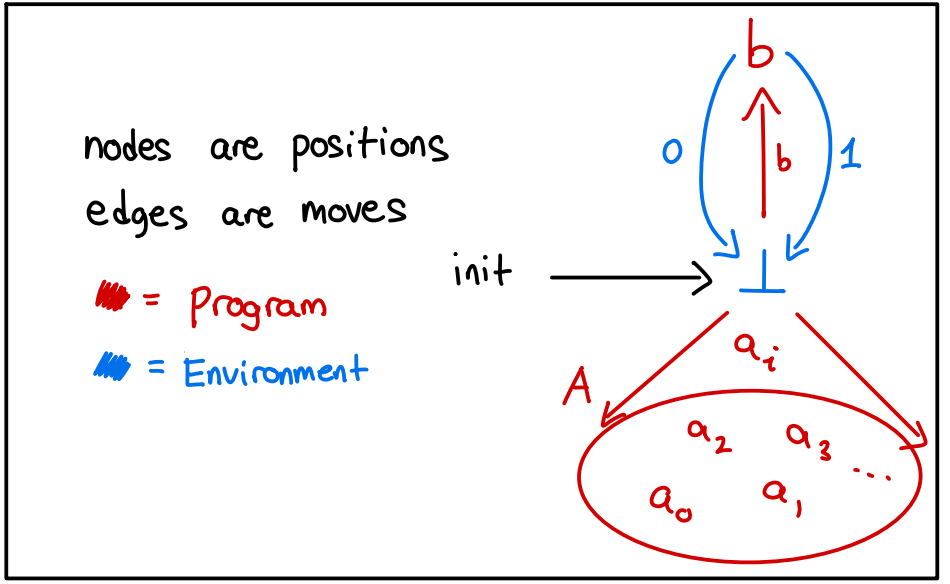
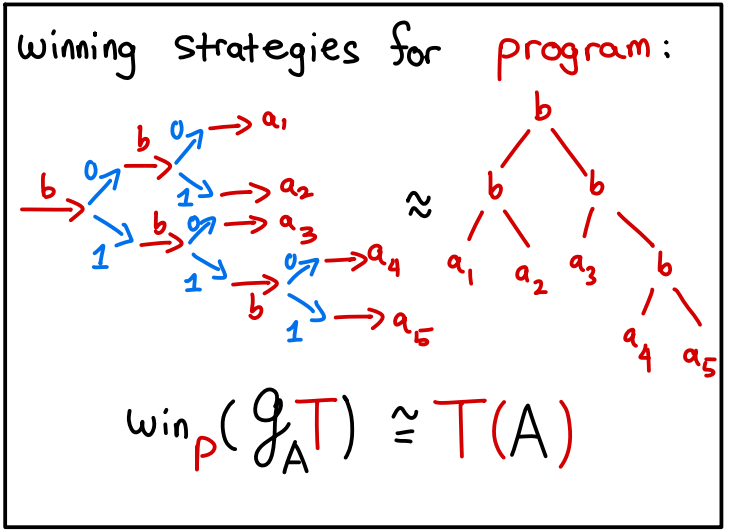
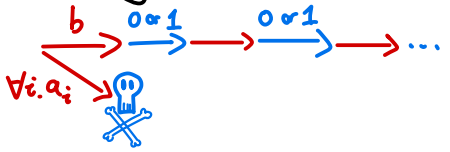




"winning" strategies for Environment:



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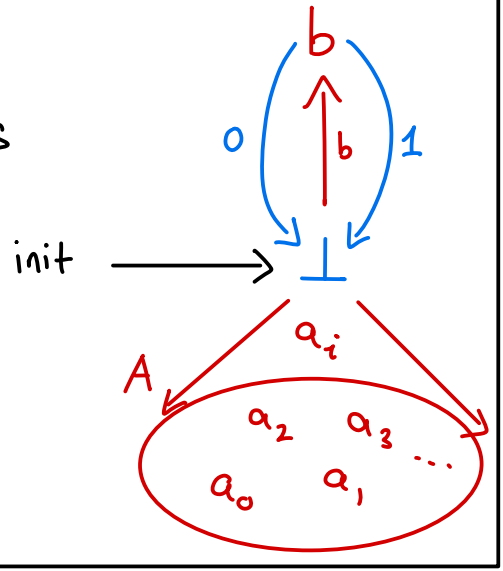

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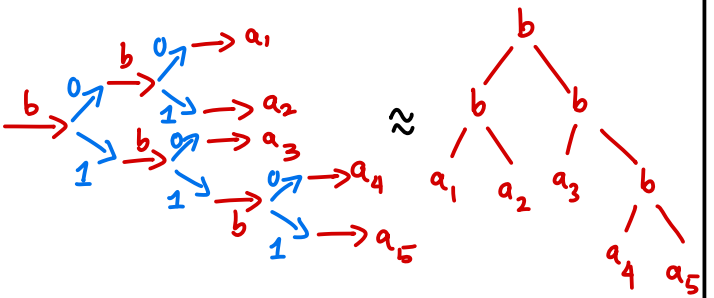

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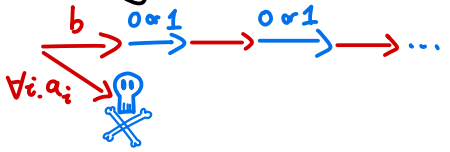

$$\mathrm{win}_E(\mathcal{G}_\emptyset T) \cong 2^\omega$$

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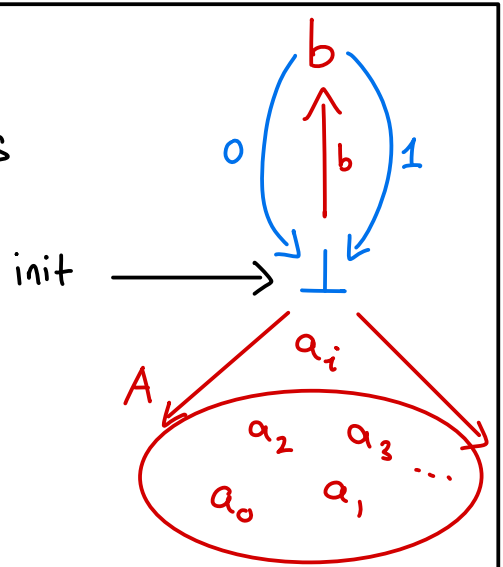

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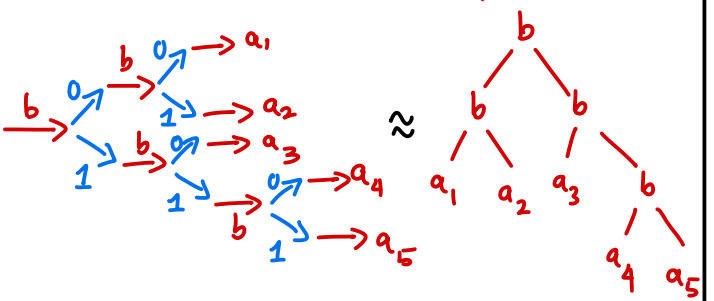

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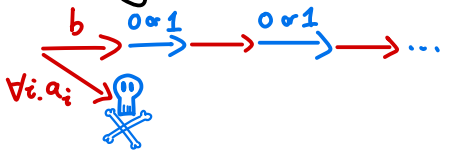

$$\mathrm{win}_E(\mathcal{G}_\emptyset T) \cong 2^\omega$$

Let $\beta \in \mathrm{win}_E(\mathcal{G}_\emptyset T)$, $t \in \mathrm{win}_P(\mathcal{G}_A T)$

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Original Idea found in [Koenig 2021] but also Richard + Soichiro?

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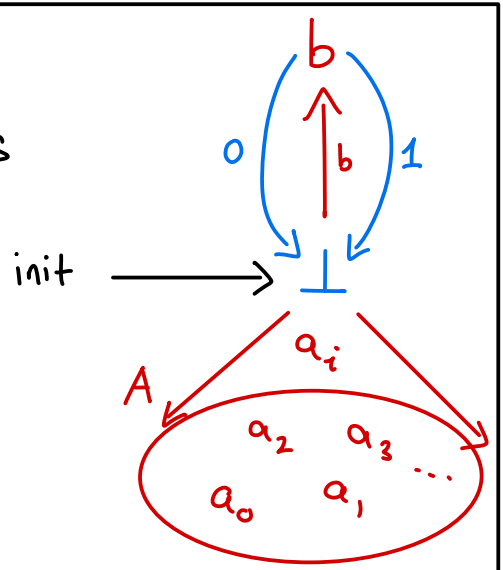

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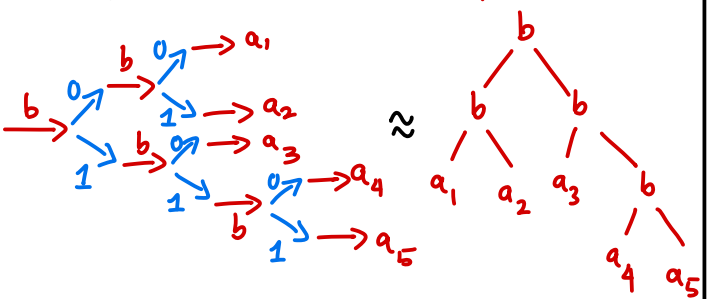

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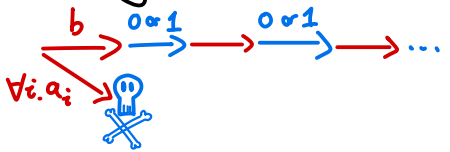


$$\mathrm{win}_E(\mathcal{G}_\emptyset T) \cong 2^\omega$$

Let $\beta \in \mathrm{win}_E(\mathcal{G}_\emptyset T)$, $t \in \mathrm{win}_P(\mathcal{G}_A T)$

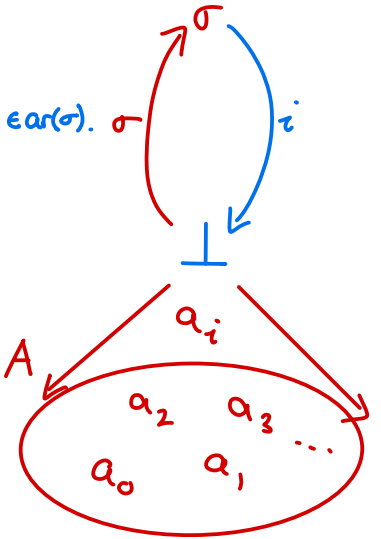
then $\langle \beta \mid t \rangle \approx ([t]_\beta, \beta(t))$

$\mathrm{Hom}_{\mathbb{BT}}(\beta, -) \cong \{\langle \beta \mid t \rangle \mid t \in T1\}$

In general, for $T = [\Sigma \mid \emptyset]$:

definition

$$q_A \Sigma = \forall \sigma . \forall i \in ar(\sigma) .\ \sigma\ i$$



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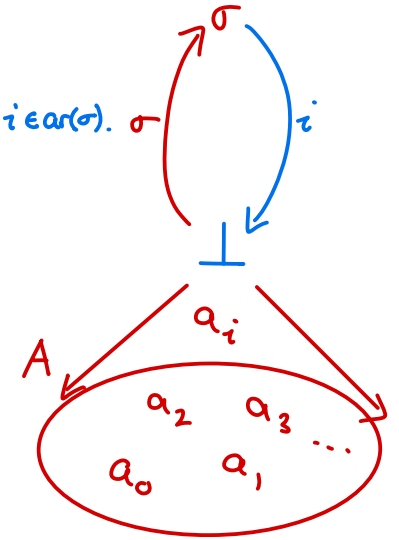

= Program

= Environment

remark if $z/0 \in \Sigma$ then $win_E(\mathcal{G}_\emptyset \Sigma) \cong \emptyset$.
but $\mathcal{G}_\emptyset \Sigma$ is not trivial!

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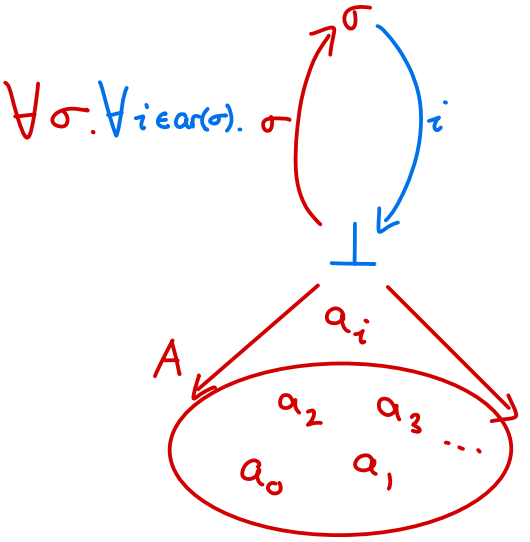
$$\mathcal{G}_A\Sigma =$$



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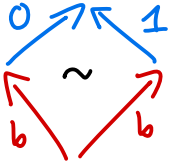
question What about $[b/2 \mid b(x,y) = b(y,x)]$?

Intuitively, want to add 2-cell:

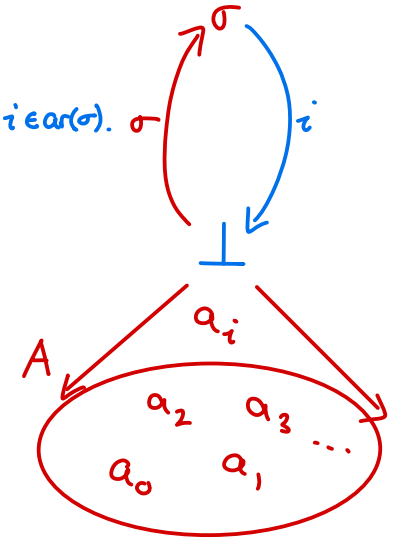


$$\{\beta \in \mathrm{win}_E(\mathcal{G}_\emptyset b) \mid \beta \sim \beta\} = \emptyset$$

In general, for $T = [\Sigma \mid \emptyset]$:

definition

$$g_A\Sigma = \quad \forall \sigma.\forall_{i \in ar(\sigma)}.$$



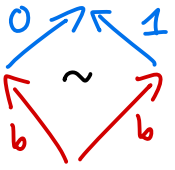
= Program

= Environment

remark if $z/0 \in \Sigma$ then $win_E(g_\emptyset\Sigma) \cong \emptyset$.
but $g_\emptyset\Sigma$ is not trivial!

question What about $[b/2 \mid b(x,y) = b(y,x)]$?

Intuitively, want to add 2-cell:



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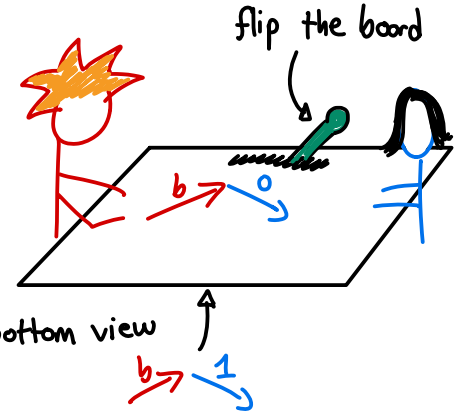

The most "useless" definition of games

"template"

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$$\$ames(\mathbb{T}) = \$pan(\mathcal{C})/_{\mathbb{T}}$$

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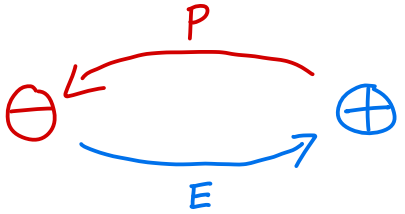
$\mathbb{T}_\circ$ (freely generated by)



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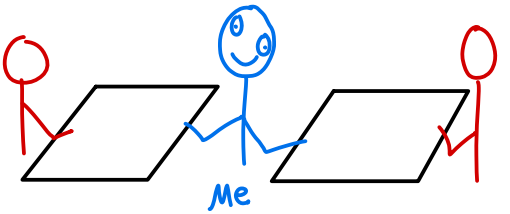

"war on two fronts"

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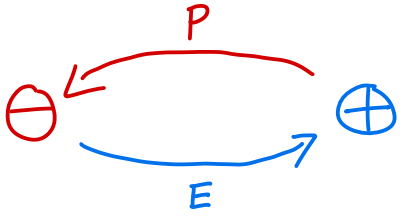
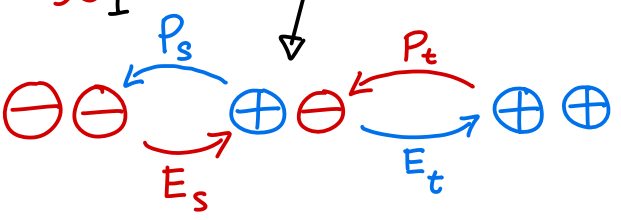


$\mathbb{T}_1$



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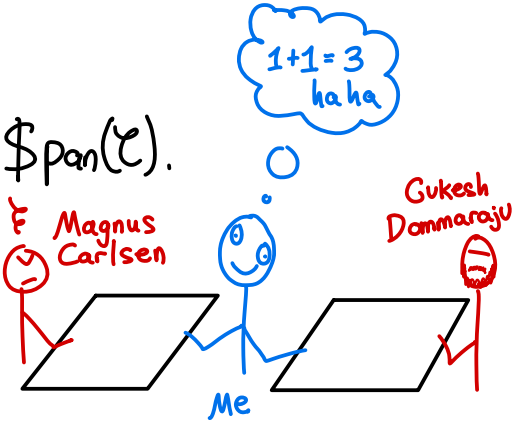

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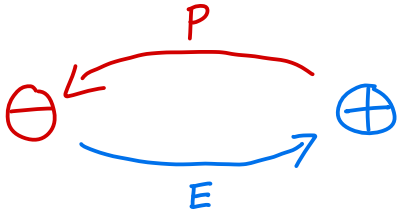

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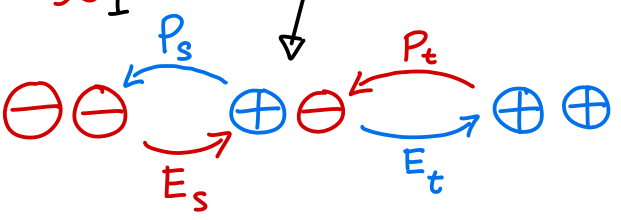

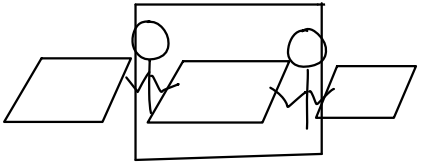

Parallel Composition + hiding

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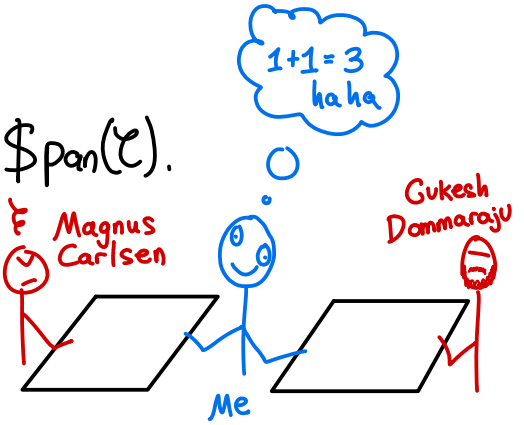

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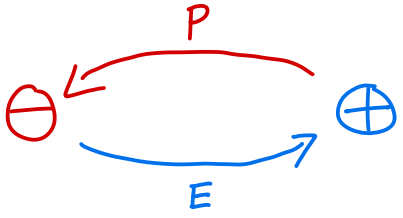

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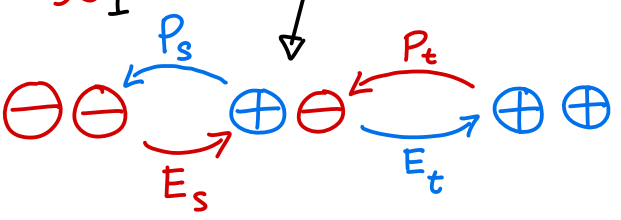

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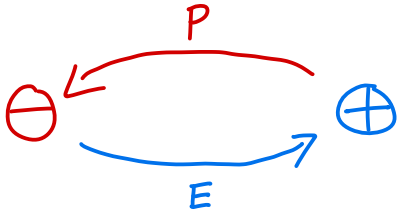

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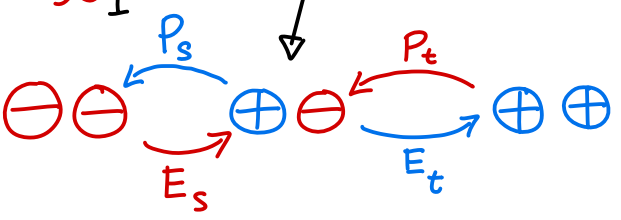

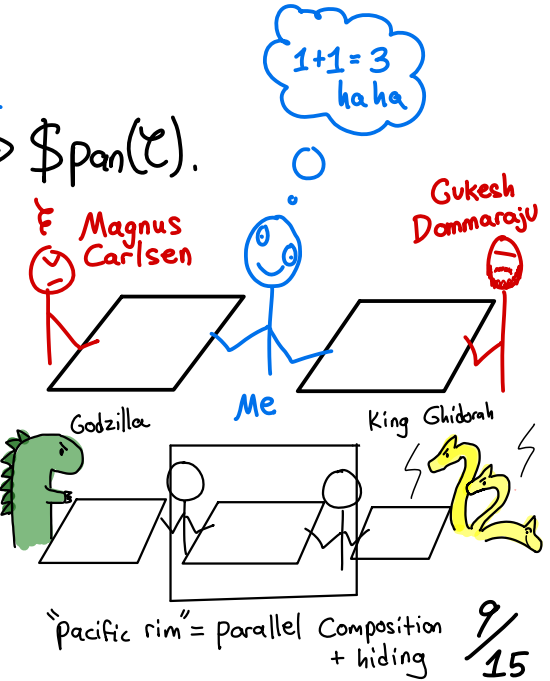

Less "useless" definition

- objects are games $\left(G \in \mathrm{Cat}, G \xrightarrow{\lambda_G} \not{\pi}_0\right)$

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- tight maps are reductions $G \xrightarrow{f} G'$ (commuting over $\mathbb{t}_0$)
- loose maps are strategies $G_1 \xrightarrow{(S,\lambda,s,t)} G_2$

$$\begin{array}{ccccc} G_1 & \xleftarrow{s} & S & \xrightarrow{t} & G_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{t}_0 & \longleftarrow & \mathbb{t}_1 & \longrightarrow & \mathbb{t}_0 \end{array}$$

Less "useless" definition

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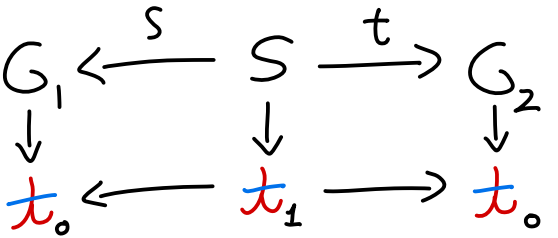
Note that this is very abstract and doesn't demand the usual constraints (winning, deterministic, responsive)



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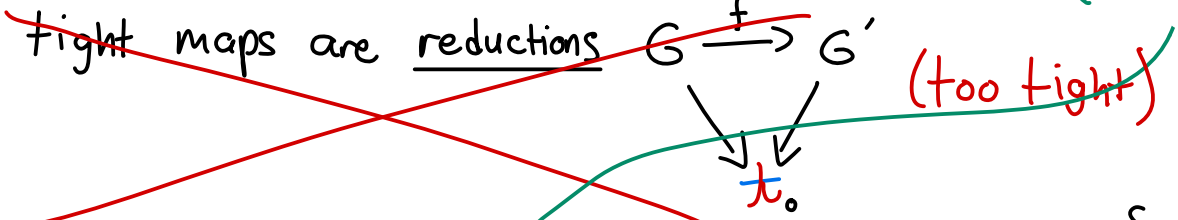
- ~~tight maps are reductions~~ $G \xrightarrow{f} G'$ (too tight)



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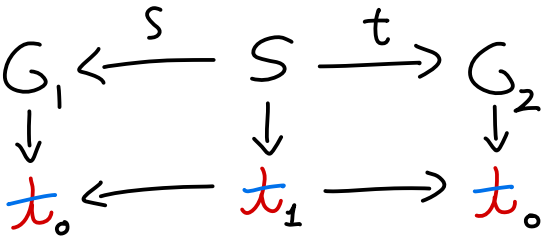
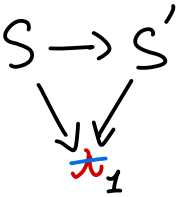


- 2-cells are



making bla bla bla commute.

$\mathbb{G}\mathrm{ames}(\mathbb{t})$ bicategory of template games.

We want to define a map

$$\text{Signatures} \xrightarrow{\mathcal{G}} \text{Games}(\not{\lambda})$$

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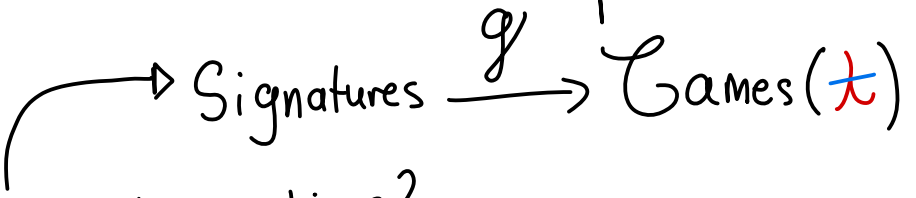

$[\Sigma_1 | \emptyset] \xrightarrow{\varphi} [\Sigma_2 | \emptyset]$ monad map

We want to define a map

$$\text{Signatures} \xrightarrow{g} \text{Games}(\not{\lambda})$$

what are the morphisms?

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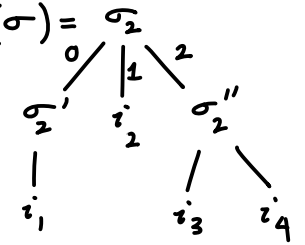
$g\,\varphi$

where (for example)

$\varphi(\sigma) =$



and $i_j \in ar(\sigma)$

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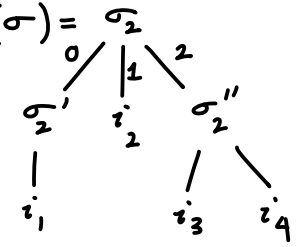

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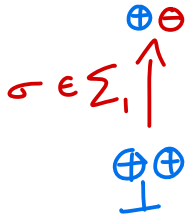

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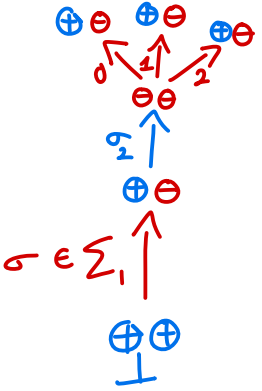

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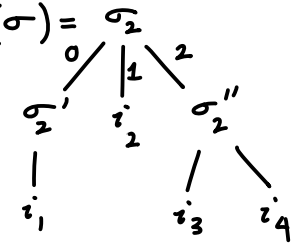

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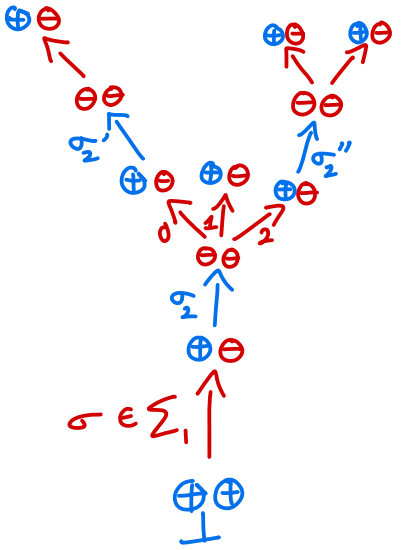

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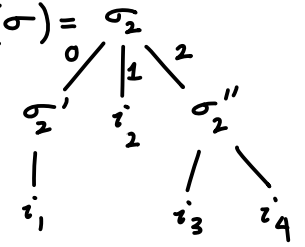

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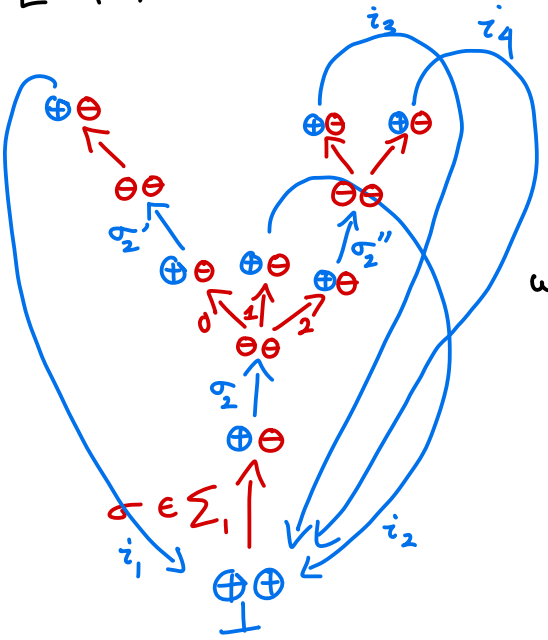

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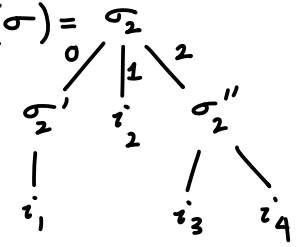

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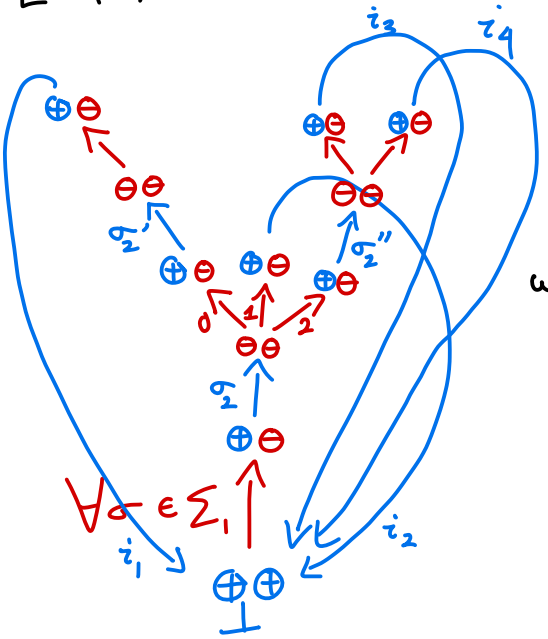

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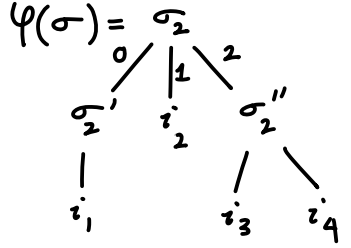
$\varphi(\sigma) =$ 

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But also:

$[\Sigma_1 | \emptyset]$-comodel (W, ρ)

where $\rho : TA \longrightarrow (W \times A)^W$

$\rho(\sigma)(w) = (w', i)$

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Signatures $\xrightarrow{\mathcal{G}}$ Games($\mathcal{T}$)

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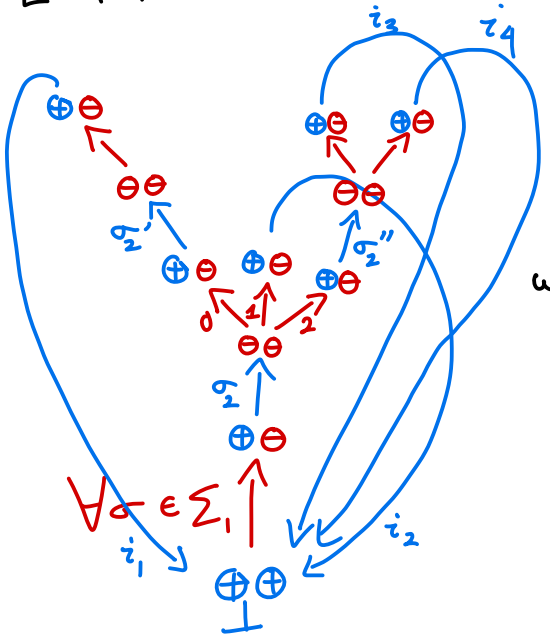

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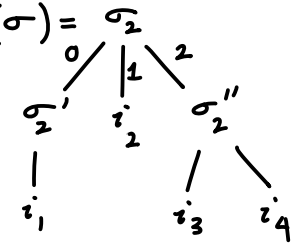
$\varphi(\sigma) =$ 

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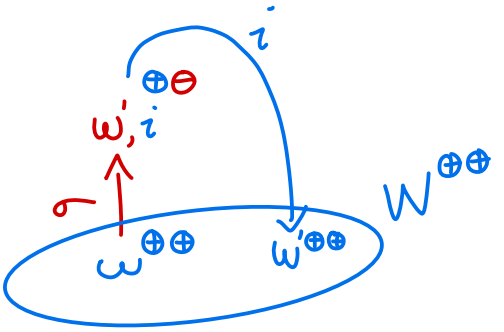

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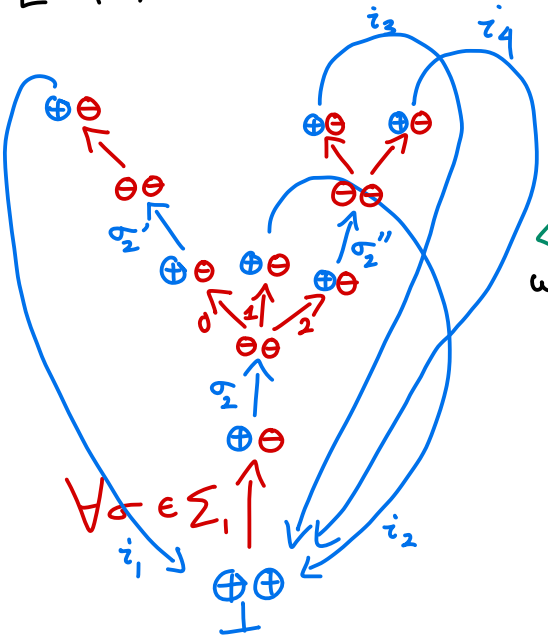

$\mathcal{G}\varphi$

a common picture!

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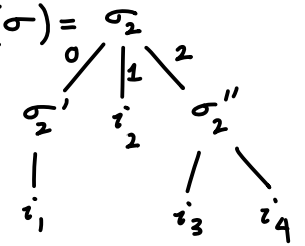
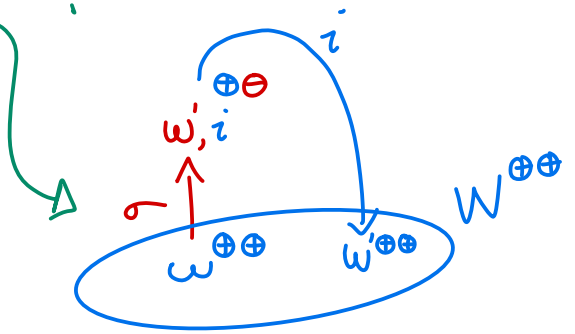


and $i_j \in ar(\sigma)$



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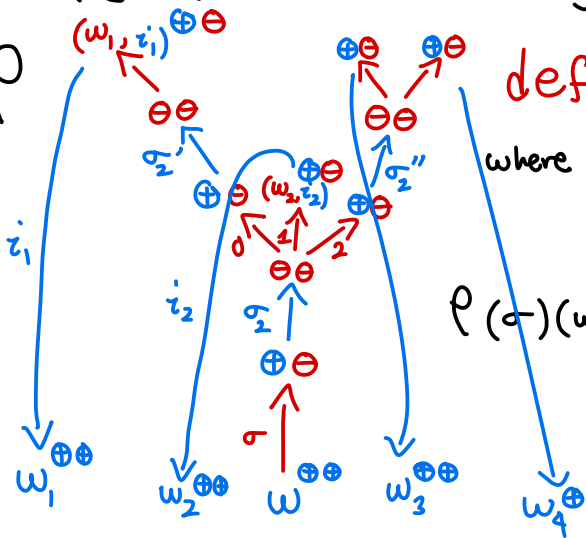
prop $\mathcal{G}\rho$  defines a pseudofunctor

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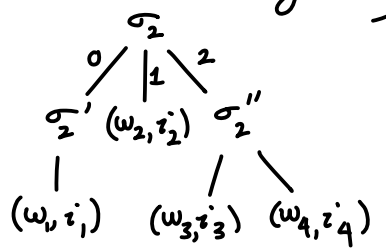
$\rho(\sigma)(\omega) =$ 

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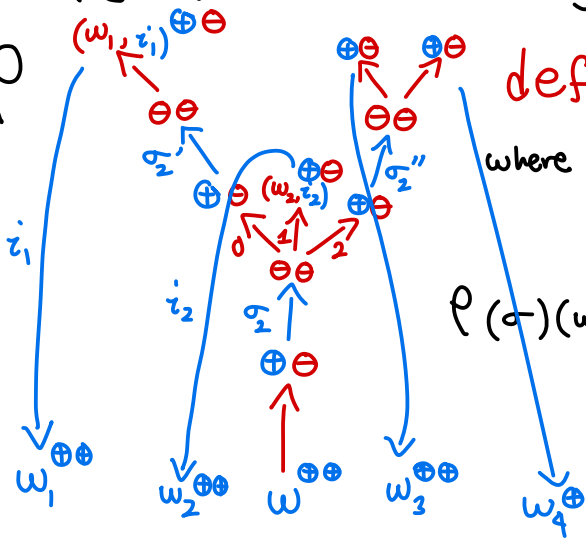
prop $\mathcal{G}\rho$ 

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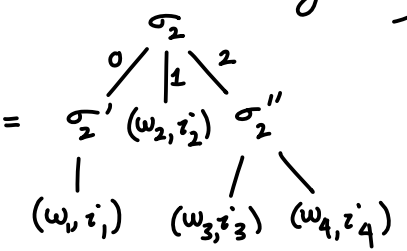
$\rho(\sigma)(\omega) =$ 

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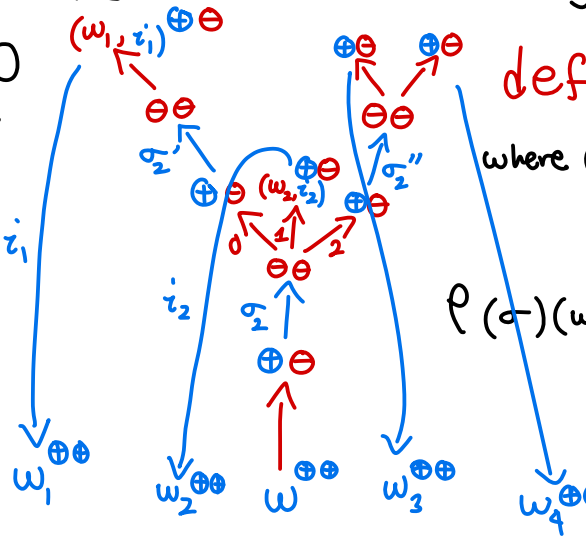

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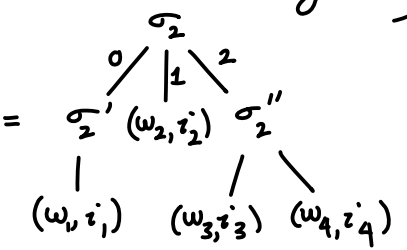

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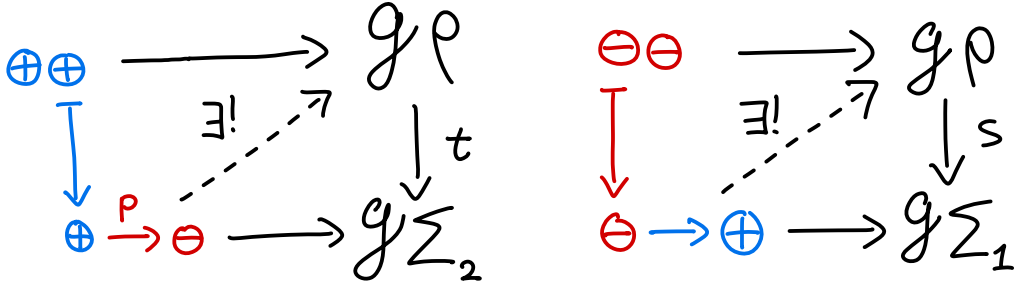

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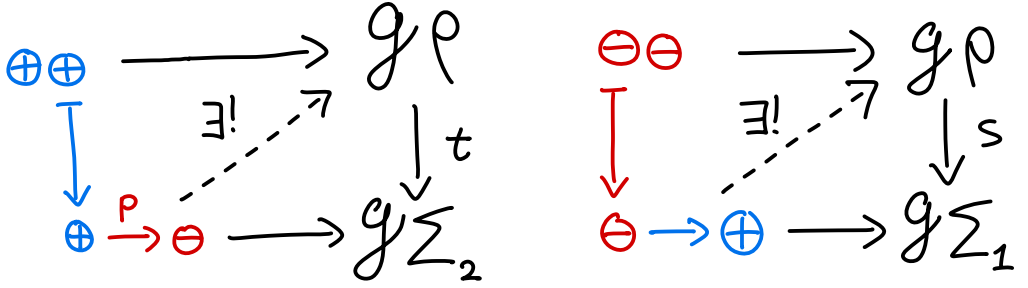

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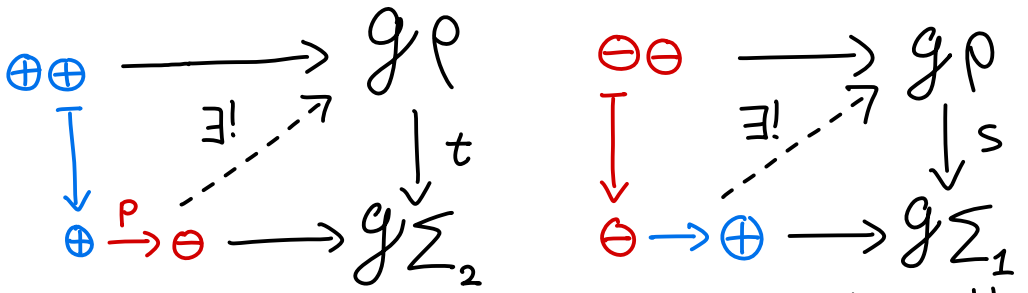

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Question What structure do we need on the LHS to get adjoint construction to $\mathcal{G}$? Statefully indexed family of signatures? 13/15

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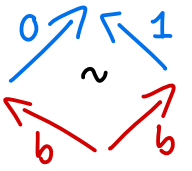

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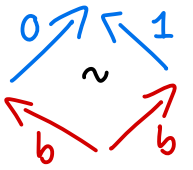

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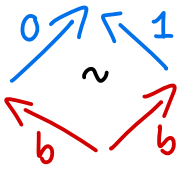

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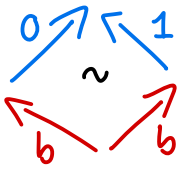

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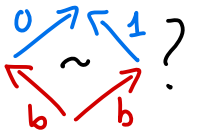

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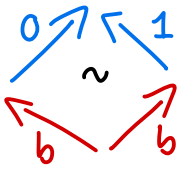

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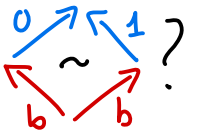

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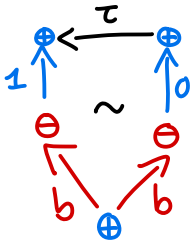

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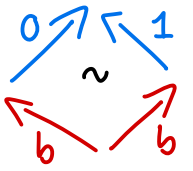

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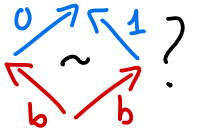

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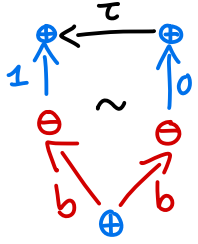

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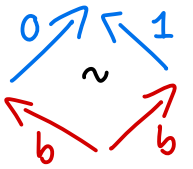

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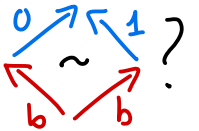

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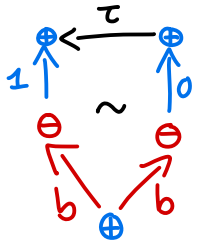

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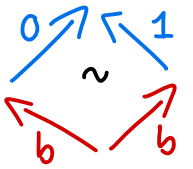

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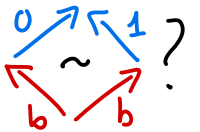

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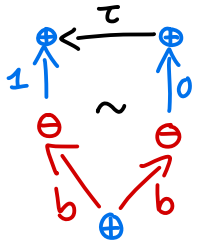

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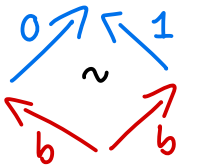

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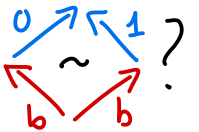

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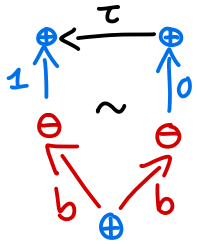

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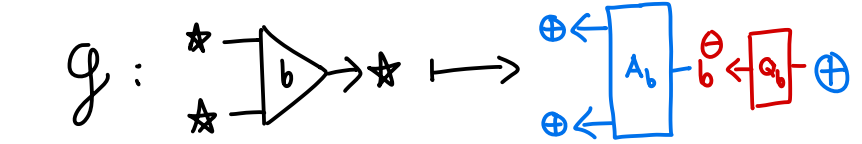

polarizes the 2-category.

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Functor of Points, sure, but Computational Meaning?

- What does a theory of game schemes look like?

And even further into the future

- Commutative AG is ultimately about solving polynomial zeros <=> polynomial eqns.

Polynomial ~ program syntax

where are $p = q$? ~ where are $P \simeq Q$? (Full Abstraction)

- Can we get presentation-invariant structures out of $\mathcal{G}[\Sigma|E]$?

BT is one example.

What about (co)homology?

- C^*-algebras are also often talked about in terms of localic groupoids. Can we get game spectra of presentations for C^*-algebras, and build a theory of non-commutative algebraic geometry? What is the relationship with Connes-style NCAG?

References

[Garner, R, Wu 2026] "Stone Duality for Monads"

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[—11— 2019] "Template Games and differential linear logic"

[—11— 2021] "Asynchronous Template Games and the Gray Tensor Product of 2-categories"

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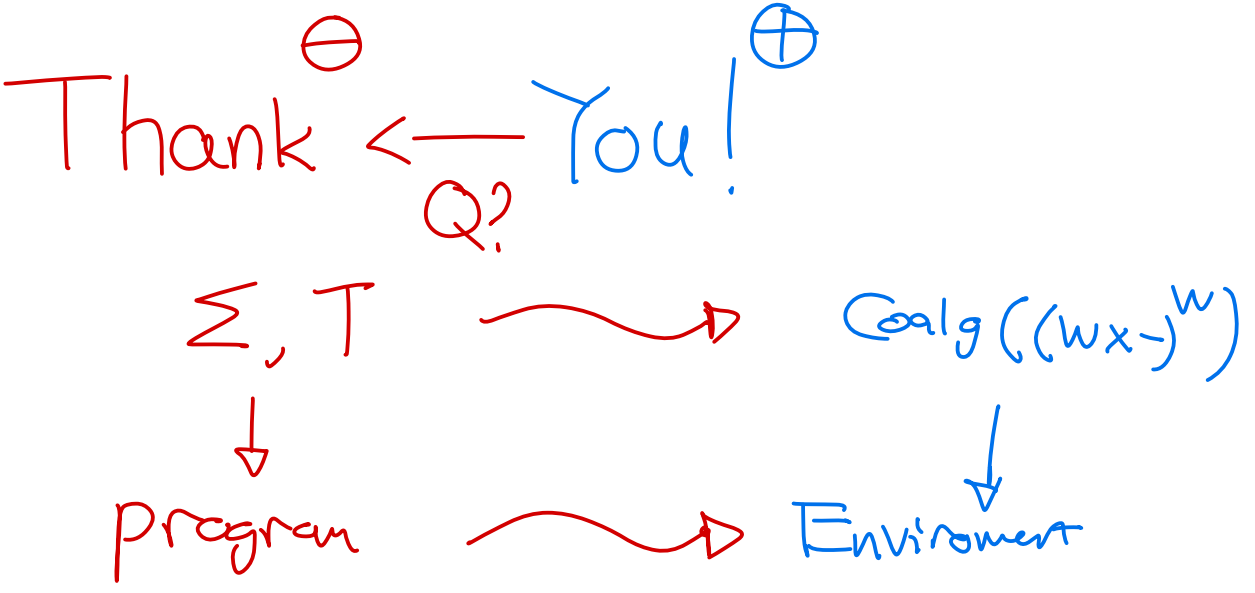
[Garner 2023] "Stream Processors and comodels"

[Calk 2025] "Higher Catoids, Higher Quantales and their correspondences"

Thank ⊖ You! ⊕


Thank ⊖
You! ⊕
Q?
Σ, T
Coalg((Wx-)^W)
program
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