From Statistical Data to Geometry and Logic via Isbell Nuclei

with S. Jarvis (CUNY), J.-L. Galstaldi (ETH Zurich), J. Terilla (CUNY)

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https://www.seiller.org/documents/PCTseminar.pdf

Motivation: textual corpora

Question

What structure of language hides in statistical data on a textual corpus?

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What structure of language hides in statistical data on a textual corpus?

- LLMs as witnesses that such a structure exist (though corpus-dependent)
- Some results in that direction about word embeddings (Lévy et Goldberg, 2014)

Word embeddings and SVD

Word embeddings

Deep neural network: a function $f : \mathbf{R}^{n_0} \to \mathbf{R}^{n_k}$ explicitly expressed as a composition

$$\mathbf{R}^{n_0} \xrightarrow{f_1} \mathbf{R}^{n_1} \xrightarrow{f_2} \mathbf{R}^{n_2} \xrightarrow{f_3} \dots \xrightarrow{f_{k-1}} \mathbf{R}^{n_{k-1}} \xrightarrow{f_k} \mathbf{R}^{n_k}$$

where

$$f_i(x) = a(M_i x + b_i)$$

where M_i are $n_i \times n_i$ matrices, $b_i \in \mathbf{R}^{n_i}$ are *biases*, a is a (nonlinear) *activation* function, and g is an output function.

- Quick realisation: significant increase of performance when the first layer (i.e. f_1) of a model trained for a linguistic task is used as first layer in another model aimed at a different linguistic task.
- People started training these *word embeddings* separately.

Motivation: textual corpora

- Word embedding capture semantic information. E.g. the vector for `Berlin` minus the vector for `Germany` is numerically very near the vector for `Paris` minus the vector for `France`.

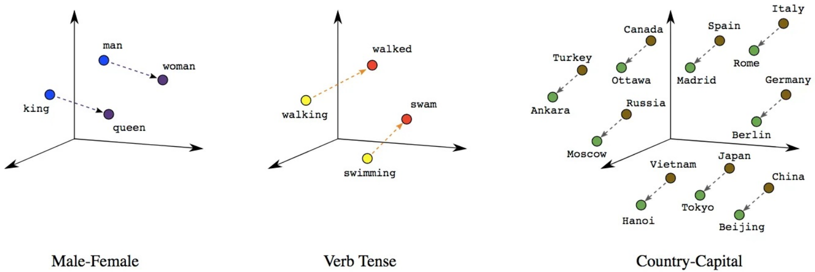

Figure: Image from developers.google.com

- Levy and Goldberg: word embeddings perform an implicit factorization of a matrix comprised of information about how words are used in language.

Motivation: textual corpora

Lévy et Goldberg (2014): word embeddings as implicit matrix factorization.

John has kids. Thomas has kids. Thomas is french. Samantha made popcorn icecream. Thomas does not live in NYC. Gianni does not live in NYC. John made popcorn icecream. Thomas made popcorn icecream.

Consider the matrix *pmi* ($\log(p(ab)/p(a)p(b))$; here simplified).

	has kids	is french	made icecream	not in NYC
John	$\log\left(\frac{16}{2} \times \frac{16}{2} \times \frac{1}{15}\right)$	$-\infty$	$\log\left(\frac{16}{2} \times \frac{16}{2} \times \frac{1}{15}\right)$	$-\infty$
Thomas	$\log\left(\frac{16}{4} \times \frac{16}{2} \times \frac{1}{15}\right)$	$\log\left(\frac{16}{4} \times \frac{16}{2} \times \frac{1}{15}\right)$	$\log\left(\frac{16}{4} \times \frac{16}{2} \times \frac{1}{15}\right)$	$\log\left(\frac{16}{4} \times \frac{16}{2} \times \frac{1}{15}\right)$
Gianni	$-\infty$	$-\infty$	$-\infty$	$\log\left(\frac{16}{1} \times \frac{16}{2} \times \frac{1}{15}\right)$
Samantha	$-\infty$	$-\infty$	$\log\left(\frac{16}{1} \times \frac{16}{2} \times \frac{1}{15}\right)$	$-\infty$

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	has kids	is french	made icecream	not in NYC
John	1.36991	$-\infty$	1.36991	$-\infty$
Thomas	1.67094	1.67094	1.67094	$-\infty$
Gianni	$-\infty$	$-\infty$	$-\infty$	1.06888
Samantha	$-\infty$	$-\infty$	1.06888	$-\infty$

Motivation: textual corpora

Lévy et Goldberg (2014): word embeddings as implicit matrix factorization.

We identify the matrix as a linear operator, and approach it (SVD).

$$W \times C \xrightarrow{\;M\;} \mathbf{R} \qquad \mathbf{R}^W \underset{M_*}{\overset{M^*}{\rightleftarrows}} \mathbf{R}^C$$

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$$W \times C \xrightarrow{M = U\Sigma V^*} \mathbf{R}$$

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots \\ 0 & \lambda_2 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_W & \dots \end{pmatrix}$$

$$\mathbf{R}^W \underset{U_*}{\overset{U^*}{\rightleftarrows}} \mathbf{R}^W \underset{\Sigma_*}{\overset{\Sigma^*}{\rightleftarrows}} \mathbf{R}^C \underset{{V^t}_*}{\overset{{V^t}^*}{\rightleftarrows}} \mathbf{R}^C$$

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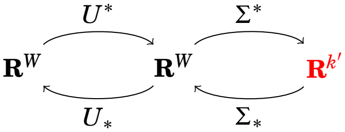
We identify the matrix as a linear operator, and approach it (SVD).

$$W \times k \xrightarrow{M = U\tilde{\Sigma}} \mathbf{R}$$

$$\tilde{\Sigma} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots \\ 0 & \lambda_2 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_k & \dots \end{pmatrix}$$

$$k << W$$



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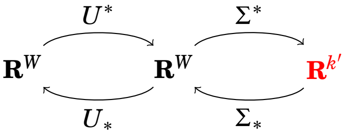
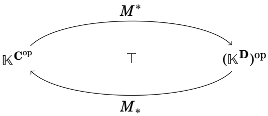


Objective 1: proceed similarly, using a natural structure (addition, order).
I.e. avoid imposing that of linear map – considering addition and multiplication.
Objective 2: compositionality.

The nucleus

Nucleus of enriched adjunctions

Start from $M : \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbb{K}$.



Nucleus: fixed points (a,b) such that $M^*(a) = b$ and $M_*(b) = a$

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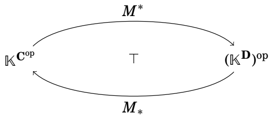

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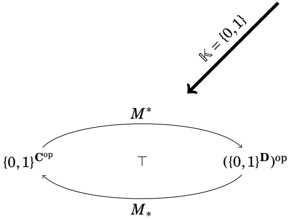

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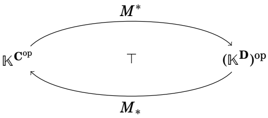

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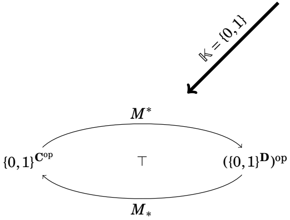

Nucleus: formal concepts

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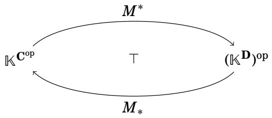

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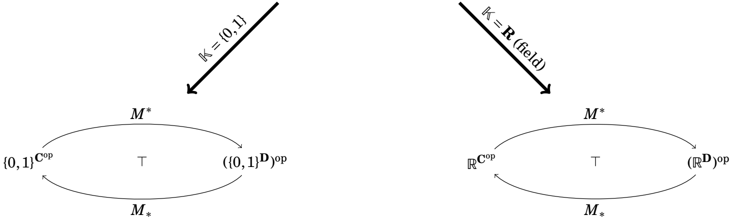

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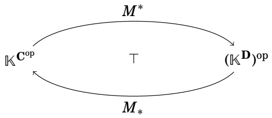

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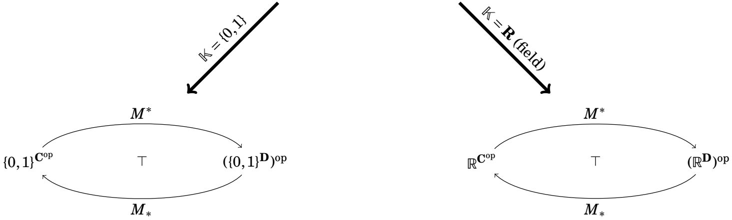

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Nucleus: singular vectors (SVD)

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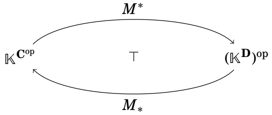

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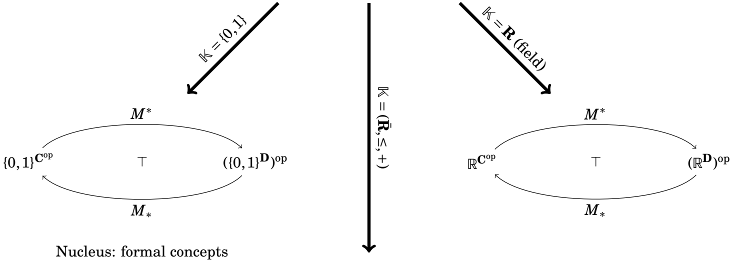

Nucleus: formal concepts

This talk

Background: formal concepts

Formal concepts

Consider for a moment that instead of a **Set**-valued profunctor, we start with

$$M : \mathbf{X}^{\mathrm{op}} \times \mathbf{Y} \rightarrow \mathbf{2},$$

where $\mathbf{2}$ is the category $0 \leftarrow 1$, and $\mathbf{X}, \mathbf{Y}$ are finite set (seen as discrete categories).

- The profunctor M is simply a binary relation;
- The maps M_* and M^* are defined on *subsets* of $\mathbf{C}$ and $\mathbf{D}$. More precisely, if $A \subset \mathbf{X}$ and $B \subset \mathbf{Y}$:

$$M^*(A) = \{b \in \mathbf{Y} \mid \forall a \in A, M(a,b) = 1\}$$

$$M_*(B) = \{a \in \mathbf{X} \mid \forall b \in B, M(b,a) = 1\}$$

- The elements of the *nucleus* of the adjunction, i.e. pairs of subsets (A,B) such that $M^*(A) = B$ and $M_*(B) = A$ are *formal concepts*.

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$$M^*(A) = \{b \in \mathbf{Y} \mid \forall a \in A, a\,M\,b\} = A^M$$

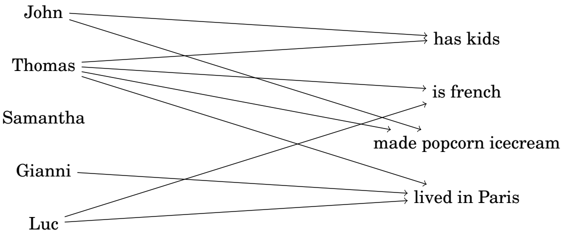
$$M_*(B) = \{a \in \mathbf{X} \mid \forall b \in B, a\,M\,b\} = {}^M B$$

- The elements of the *nucleus* of the adjunction, i.e. pairs of subsets (A, B) such that $M^*(A) = B$ and $M_*(B) = A$ are *formal concepts*.

Formal concepts

Formal concepts analysis is a method building on work by Birkhoff in which a lattice of *formal concepts* is derived from a binary relation between two sets. It has been used in computer science to define formal ontologies.

Suppose I am given the following information.



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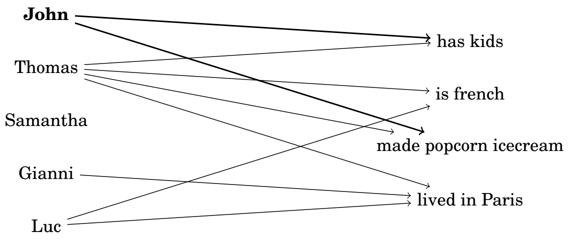

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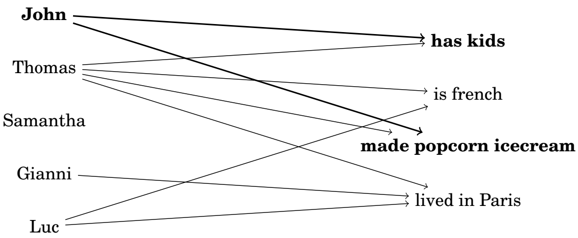

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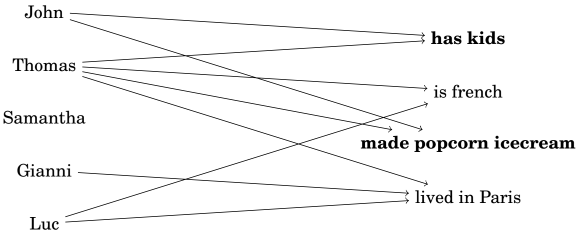

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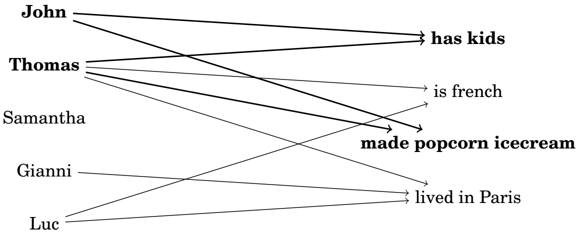

Formal concepts

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- Start from a binary relation $R \subset X \times Y$.
- Given a subset $A \subset X$, define $A^R = \{y \mid \forall a \in A, a R y\}$.
- Given a subset $B \subset Y$, define ${}^R B = \{x \mid \forall b \in B, x R b\}$.
- A formal concept is a set $A = {}^R B$.
- Equivalently, it is a set A such that ${}^R(A^R) = A$.
- The map $A \mapsto {}^R(A^R)$ is a closure operator.

The nucleus, geometrically

Projective metric geometry of tropical nuclei: gap matrices, event loci, and order chambers
with J.-L. Gastaldi, S. Jarvis, J. Terilla, April 2026
https://hal.archives-ouvertes.fr/hal-05452748

Nucleus

The Isbell nucleus is defined for any $\bar{\mathbf{R}}$-enriched profunctor

$$M : \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \bar{\mathbf{R}},$$

where $\bar{\mathbf{R}}$ is the monoidal poset $\bar{\mathbf{R}} = ([-\infty, \infty], \leq, +)$.
The Isbell conjugates M^* and M_* form an adjunction between $\bar{\mathbf{R}}$-enriched presheaves on $\mathbf{C}$ and copresheaves on $\mathbf{D}$:

$$M^* f(d) = \min_{c \in \mathbf{C}} \big(M(c,d) - f(c) \big), \qquad M_* g(c) = \min_{d \in \mathbf{D}} \big(M(c,d) - g(d) \big).$$

The nucleus $\mathrm{Nuc}(M)$ is the fixed-point locus, the set of pairs (f, g) with $M^* f = g$ and $M_* g = f$.

First geometric structures

There is a gauge action ($\lambda \in \mathbf{R}$):

$$(f,g) \in \mathrm{Nuc}(M) \Rightarrow (f+\lambda, g-\lambda) \in \mathrm{Nuc}(M).$$

When **C** and **D** are finite sets and M is a real matrix, the projectivization $\mathbb{P}\mathrm{Nuc}(M)$ is a compact polyhedral space carrying two canonical structures:

- a *projective metric* coming from the enriched hom, and
- a *polyhedral cell decomposition* coming from the Isbell inequalities $f(c)+g(d) \leq M(c,d)$; a point is classified by the incidences (c,d) for which equality holds, which we call *witness pairs*.

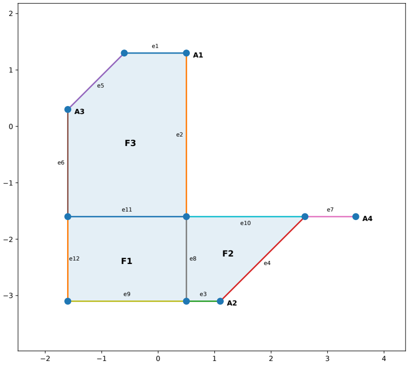
Both structures are invariant under external gauge transformations – reweighting M by row and column potentials.

The nucleus is the intrinsic geometric object; the matrix M is just a coordinate presentation.

Running example

Let $C = \{c_0, c_1, c_2\}$ and $D = \{d_1, d_2, d_3, d_4\}$ and set

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2.0 & -1.6 & 2.0 & -2.9 \end{bmatrix}.$$



Gap matrix

The central new tool is the *gap matrix* defined at each nucleus point (f,g) by

$$\delta^{(f,g)}(c,d) = M(c,d) - f(c) - g(d).$$

It is a non-negative matrix with at least one zero in every row and column. Its zero entries record the witnesses and hence the combinatorial cell. Its positive entries, which measure the slack in each inequality, turn out to have a sharp geometric meaning.

Gap matrix

In our example, $\mathbb{P}\mathrm{Nuc}(M)$ is a two-dimensional polyhedral complex. We work in the gauge slice $f(c_0) = 0$ and consider the point (f, g) with

$$f = (0, 0, 0), \qquad g = (0.7, -1.6, 0.1, -2.9).$$

Its gap matrix is

$$\delta^{(f,g)} = \begin{bmatrix} 0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0 & 5.1 \\ 1.3 & 0 & 1.9 & 0 \end{bmatrix},$$

whose positive values satisfy

$$0 < 0.5 < 1.3 < 1.6 = 1.6 < 1.9 < 3.1 < 4.2 < 5.1.$$

Events Theorem

Theorem

Each positive entry of the gap matrix equals the exact projective distance to the corresponding event locus:

$$d_{\mathbb{P}\mathrm{Nuc}}\big((f,g),\mathscr{E}_{c,d}\big) = \delta^{(f,g)}(c,d),$$

where $\mathscr{E}_{c,d}$ is the locus on which (c,d) is a witness pair.

In other words, algebraic slack in the Isbell inequalities *is* geometric distance to the cell walls. This identity reflects a rigidity special to the Hilbert metric and the linear structure of the Isbell conditions: in a generic polyhedral-metric setting, constraint slack and boundary distance can differ by arbitrary distortion factors.

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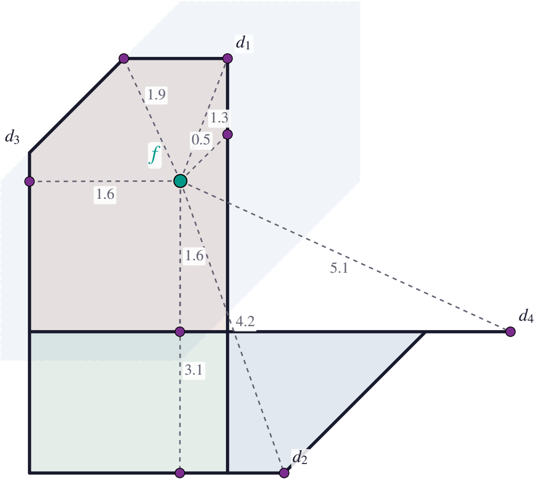
The zero pattern $\{(c_0,d_1),(c_2,d_2),(c_1,d_3),(c_2,d_4)\}$ determines the witness cell containing (f,g), while the positive entries are the exact projective distances from (f,g) to the surrounding cell walls. The tie at 1.6 is the value at which the order of the gap entries changes, producing a wall in the order-chamber decomposition.

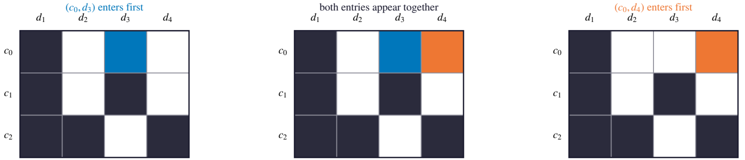
Events theorem



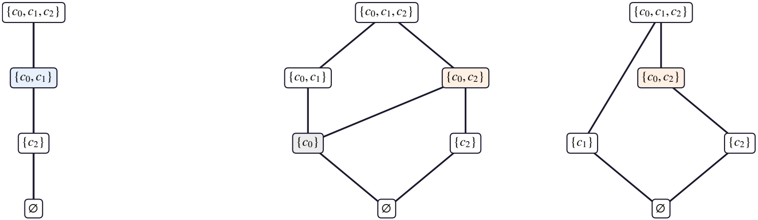
Towers of lattices

- Thresholding the gap matrix at a value $\varepsilon > 0$ records which event loci lie within projective distance ε of a given point. The resulting Boolean relation is itself a profunctor, and its Isbell nucleus is a finite lattice: a formal concept lattice.
- As ε increases, new witness pairs enter and the lattice grows. Since the gap matrix has finitely many distinct values, this growth factors through a finite tower of concept lattices, one for each distinct gap value. Wall-crossing between consecutive floors is governed by canonical mergers.
- The tower depends on the basepoint (f, g), but only through the ordering of the gap entries: within each *order chamber*—a region where the ordering is constant—the tower is invariant.
- Moving to a face of the chamber complex merges consecutive floors, producing canonical specialization maps. The order chambers give a refinement of the polyhedral decomposition defined by witnesses and appears to be new in tropical geometry.

Towers of lattices

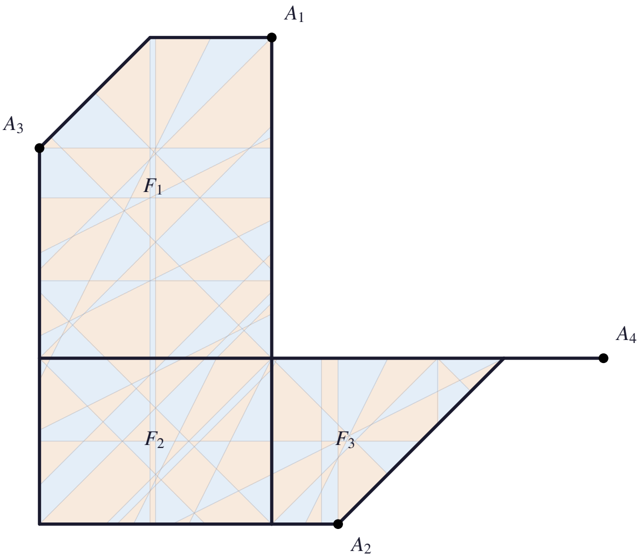


left chamber **wall** **right chamber**



$L(R_{3a})$ $L(R_4)$ $L(R_{3b})$

Order chambers



The nucleus, logically

A calculus of types in Isbell nuclei
with J.-L. Gastaldi, S. Jarvis, J. Terilla, June 2026
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Linear realisability situations

Take:

- a set P
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- $[\![\cdot,\cdot]\!]_m : P \times P \to \mathbf{R}$ a "measurement".

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Suppose moreover that the following property (trefoil property / 2-cocycle) is satisfied:

$$[\![p_1 \bullet p_2, p_3]\!]_m + [\![p_1, p_2]\!]_m = [\![p_1, p_2 \bullet p_3]\!]_m + [\![p_2, p_3]\!]_m.$$

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One can construct formal concepts (*types*) over pairs $(\alpha, a) \in \mathbf{R} \times P$, and the operation $\bullet$ can be lifted to an operation between concepts / types. This operation is a *conjunction*, with associated implication defined as $A \multimap B := (A \bullet B^{\perp})^{\perp}$.

Linear realisability situations

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- $\bullet : P \times P \to P$ an associative operation
- $[\![\cdot,\cdot]\!]_m : P \times P \to \mathbf{R}$ a "measurement".

Suppose moreover that the following property (trefoil property / 2-cocycle) is satisfied:

$$[\![p_1 \bullet p_2, p_3]\!]_m + [\![p_1, p_2]\!]_m = [\![p_1, p_2 \bullet p_3]\!]_m + [\![p_2, p_3]\!]_m.$$

One can construct formal concepts (*types*) over pairs $(\alpha, a) \in \mathbf{R} \times P$, and the operation $\bullet$ can be lifted to an operation between concepts / types. This operation is a *conjunction*, with associated implication defined as $A \multimap B := (A \bullet B^{\perp})^{\perp}$.

Special cases: geometry of interaction, ludics, interaction graphs, etc.

Interlude: PMI and the trefoil property

One key realisation was that pmi satisfies the *trefoil property* (also called the *2-cocycle property*) with respect to the concatenation of words:

$$\mathrm{pmi}(w_1 \cdot w_2, w_3)\mathrm{pmi}(w_1, w_2) = \mathrm{pmi}(w_1, w_2 \cdot w_3)\mathrm{pmi}(w_2, w_3)$$

This translates as the following property of the measure:

$$[\![w_1 \cdot w_2, w_3]\!]_m + [\![w_1, w_2]\!]_m = [\![w_1, w_2 \cdot w_3]\!]_m + [\![w_2, w_3]\!]_m.$$

This means that one can define *types* as subsets of $\mathbf{R} \times P$, following* techniques from linear realisability!

A linear realisability model

We thus follow the usual definition of linear realisability models (cf. habilitation thesis):

- Consider the set of pairs (a,α) of a word a and a real number α.
- Extend the measurement on pairs: $[\![(a,\alpha),(b,\beta)]\!]_m = \alpha+\beta-[\![a,b]\!]_m$.
- Define a binary relation: $(a,\alpha)\perp(b,\beta) \Leftrightarrow \alpha+\beta-[\![a,b]\!]_m \leq 0$.
- Define a *type* (a formal concept) as a set A such that there exists B with

$$A = {}^{\perp}B = \{(a,\alpha) \mid \forall(b,\beta)\in B, (a,\alpha)\perp(b,\beta)\}.$$

Relating the constructions

One can notice that within this specific model, the types are downward closed:

- if $\alpha.a \in A$, and $\lambda > 0$, then $(\alpha - \lambda).a \in A$.

They can therefore be understood as functions: $f_A : D \to \bar{\mathbf{R}}$, where

$$f_A(a) = \sup\{\alpha \in \mathbf{R} \mid \alpha.a \in A\}.$$

Conversely, one can transform a map $f : D \to \bar{\mathbf{R}}$ into a set of pairs:

$$A_f = \{\alpha.a \in \mathbf{R} \mid \alpha \leq f(a)\}.$$

This is a one-to-one correspondance showing the following.

Theorem

Formal concepts in the linear realisability model are exactly the elements of the nucleus of the adjunction obtained between $\mathbf{R}$*-enriched presheaves.*

Enters the logic

Every operation on the set induces an operation on formal concepts. E.g. if one defines an operation $\star : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$, it induces naturally the operation:

$$\mathbf{A} \star \mathbf{B} = {}^{\perp}(\{a \star b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp}).$$

We get an operation induced by concatenation, as well as "duals".

Definition

$$A \cdot B = {}^{\perp}(\{a \cdot b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp})$$

$$A \multimap_r B = \{f \mid \forall a \in \mathbf{A}, fa \in \mathbf{B}\}$$

$$A \multimap_l B = \{f \mid \forall a \in \mathbf{A}, af \in \mathbf{B}\}$$

Note: the situation is bit more complicated, as we need to introduce three different notions of types.

Logic and Composition

The logical approach provides an interesting new structure on the nucleus, related to composition.
We give one example of how logic describes somehow complex structural properties of the nucleus.

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- We started with $D \times D \to \mathbf{R}$.
- Now take $D = \Sigma^*$ where Σ is a base alphabet (of characters).
- In particular, it "contains" measures:

$$\underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{k} \to \mathbf{R}.$$

How are those related?

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- It induces **three** binary measure between Σ and $\Sigma \times \Sigma$.

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- Consider $(A, A^{\perp})$ in one of the corresponding nuclei; then $A^{\perp} \in \bar{\mathbf{R}}^{\Sigma \times \Sigma}$.

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- So $A^{\perp}$ is a measure on $\Sigma \times \Sigma$.

$$\mathbf{Type}(\mathit{The}) \quad \underbrace{\_\_\_\_\_\_\_\_\_}_{p(\mathit{chair})=0.1,\ p(\mathit{cat})=0.01} \quad \underbrace{\_\_\_\_\_\_\_}_{\tilde{p}}$$

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- This defines an additional complex structure of *derived nuclei*.

$$\textbf{Type}(\textit{The}) \qquad \underbrace{---------}_{p*(chair)=0,\ p*(cat)=0.9} \qquad \textbf{Type}(\textit{sleeps})$$

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- It induces **three** binary measure between Σ and $\Sigma \times \Sigma$.
- Consider $(A, A^{\perp})$ in one of the corresponding nuclei; then $A^{\perp} \in \bar{\mathbf{R}}^{\Sigma \times \Sigma}$.
- So $A^{\perp}$ is a measure on $\Sigma \times \Sigma$.
- This defines an additional complex structure of *derived nuclei*.
- It can be shown to relate to the logic structure: if (B, C) is an element of such a derived nucleus (for $A^{\perp}$), then $C = B \multimap_l A^{\perp}$ and $B = C \multimap_r A^{\perp}$.

$$\mathbf{Type}(\mathit{The}) \quad \underbrace{-\,-\,-\,-\,-\,-\,-\,-\,-}_{\mathbf{Type}(\mathit{sleeps}) \multimap \mathbf{Type}(\mathit{The})^{\perp}} \quad \mathbf{Type}(\mathit{sleeps})$$

From Statistical Data to Geometry and Logic via Isbell Nuclei

with S. Jarvis (CUNY), J.-L. Galstaldi (ETH Zurich), J. Terilla (CUNY)

Thomas Seiller
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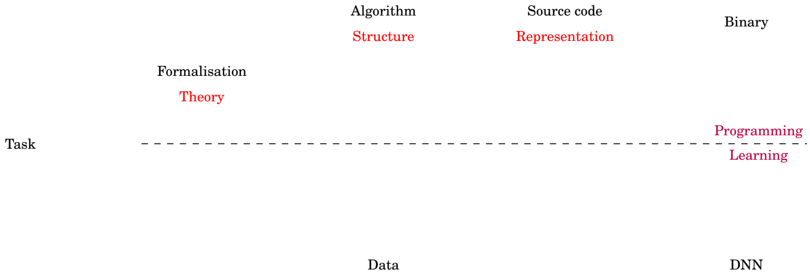
June 5th 2026

https://www.seiller.org/documents/PCTseminar.pdf

Taking a step back

Algorithms versus learning

Logic in computer science faces a challenge: it is inherently related to programming.



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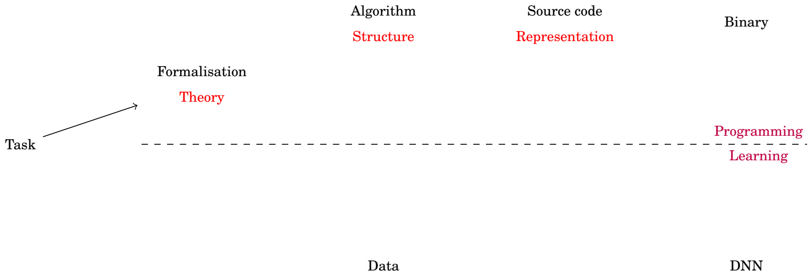

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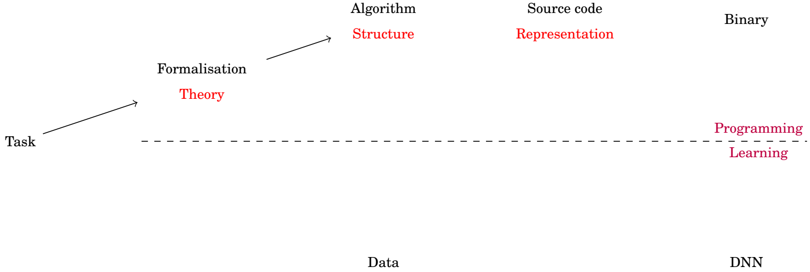

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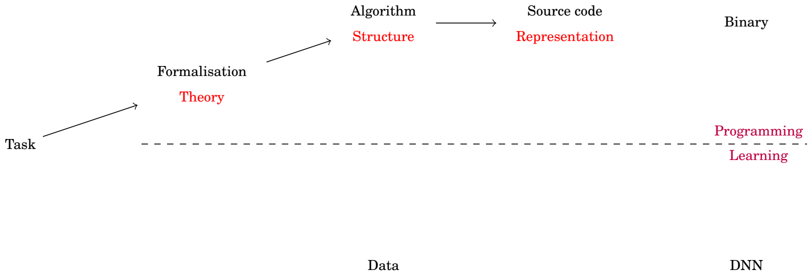

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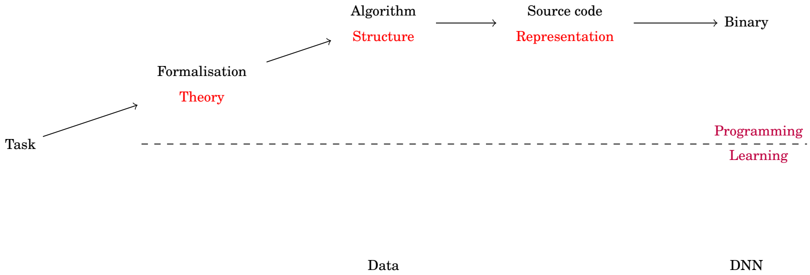

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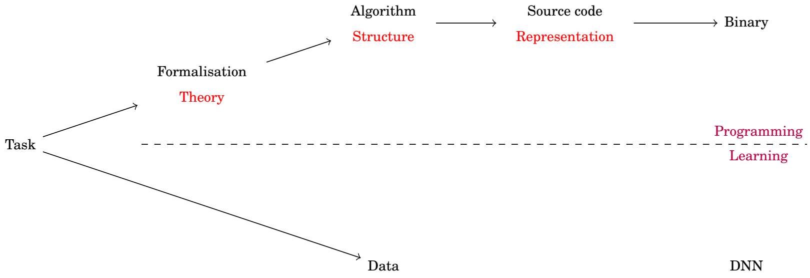

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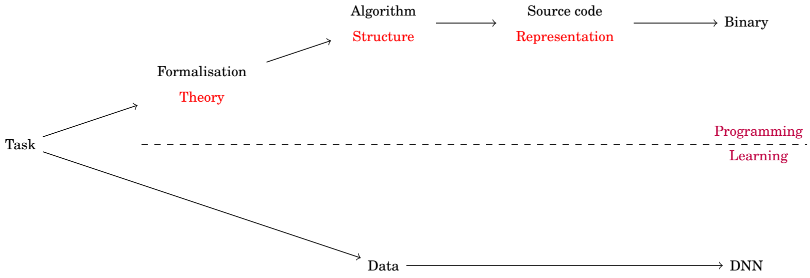

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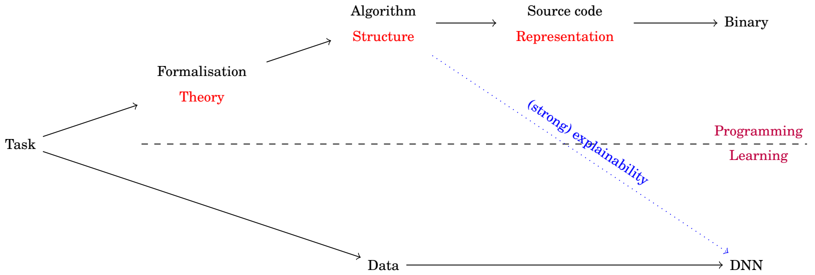

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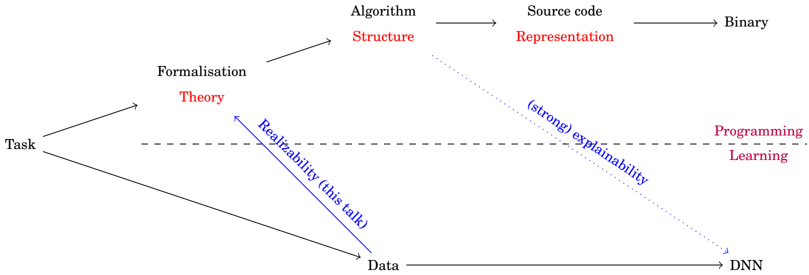
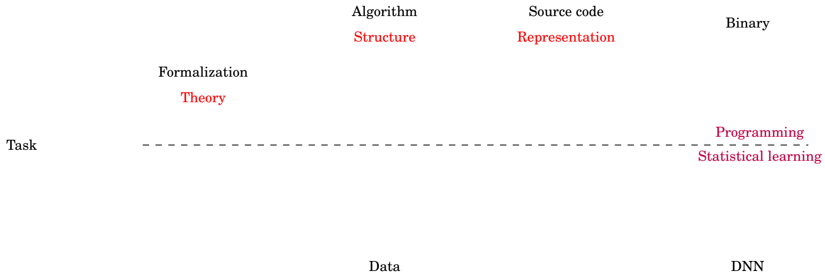


Context of the result

The interface of logic and computer science, fundamentally associated to *programming*, seems badly equipped to treat programs obtained from statistical learning.



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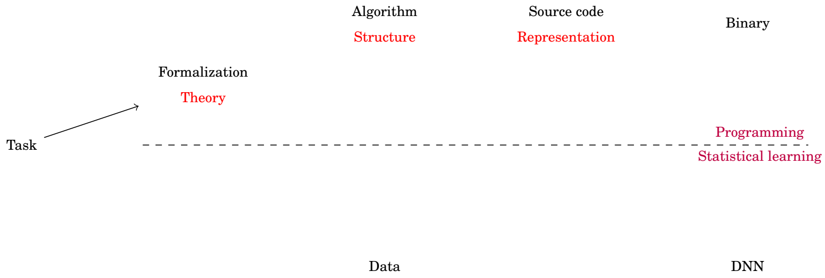

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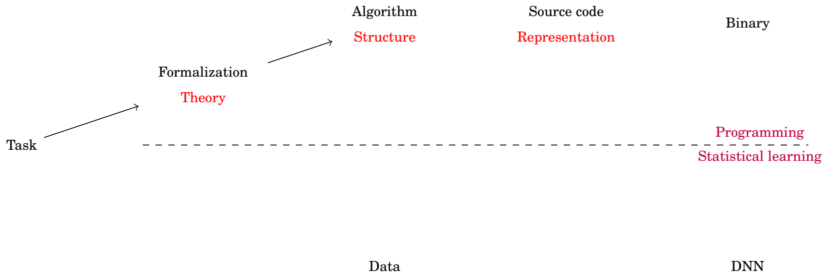

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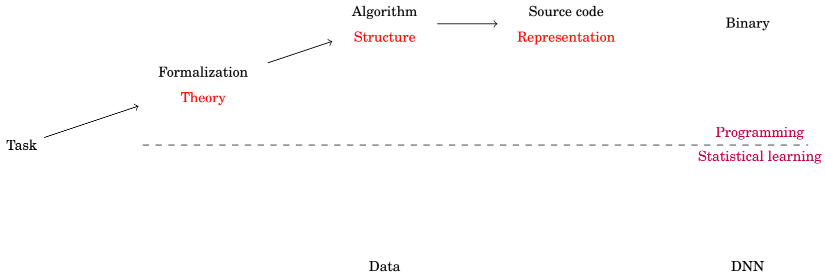

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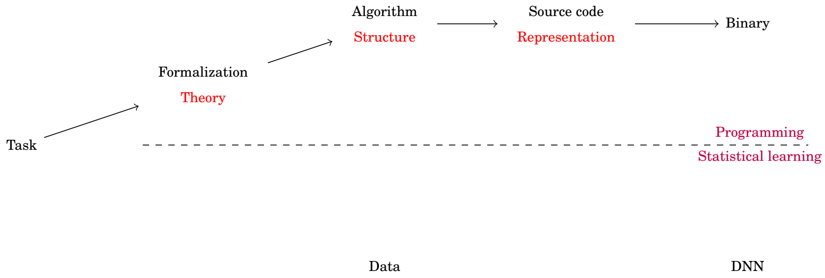

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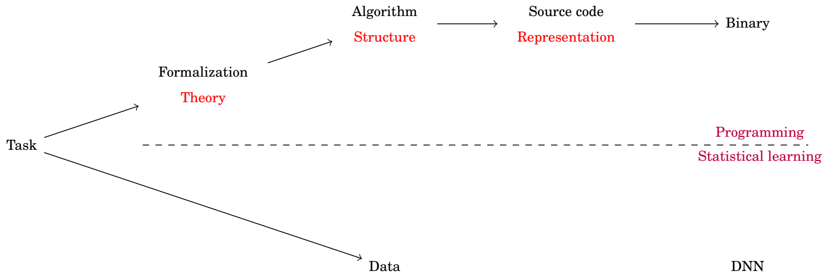

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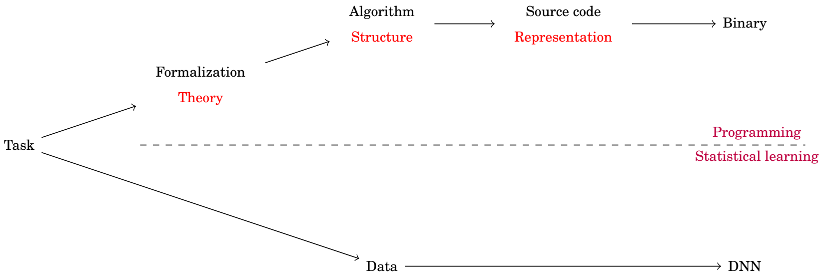

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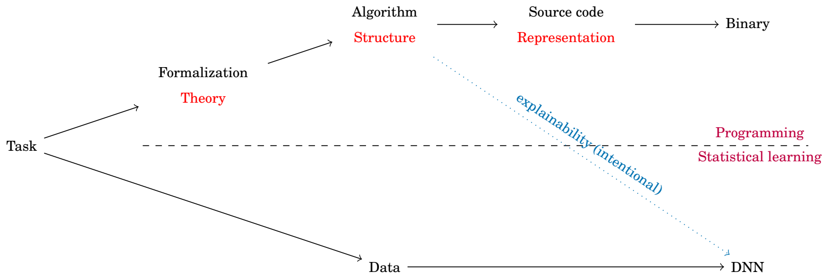

The fundamental problem is the absence of algorithms associated with programs obtained from statistical learning methods.

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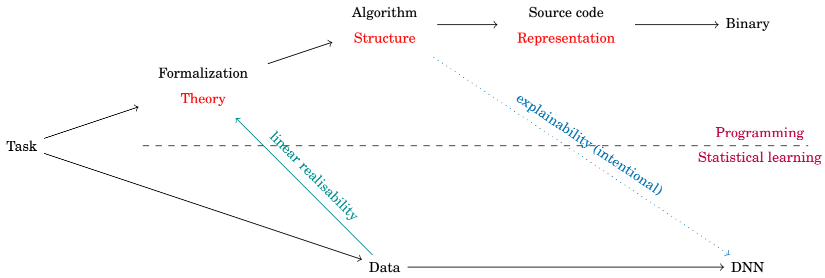

I will present a result that creates a bridge between those two worlds .

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