Towards the categorical foundation of quantale-valued sets

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Sets

A classical set X may be viewed as a set equipped with equality.

Equivalently, equality is a map

$$\alpha\colon X \times X \longrightarrow \mathbf{2} = \{0, 1\},$$

where

$$\alpha(x, y) = 1 \iff x = y.$$

Ω-sets: Sets valued in a frame

Based on the works of Higgs, Fourman–Scott proposed the notion of

Ω-set.

D. Higgs. Boolean-valued equivalence relations and complete extensions of complete boolean algebras. *Bulletin of the Australian Mathematical Society*, 3(1):65–72, 1970.

D. Higgs. Injectivity in the topos of complete Heyting algebra valued sets. *Canadian Journal of Mathematics*, 36(3):550–568, 1984.

M. P. Fourman and D. S. Scott. Sheaves and logic. *Lecture Notes in Mathematics*, 753:302–401, 1979.

Ω-sets: Sets valued in a frame

Let Ω be a frame. An Ω-set is a set X equipped with a map

$$\alpha\colon X \times X \longrightarrow \Omega$$

such that

1. $\alpha(x, y) = \alpha(y, x)$,
2. $\alpha(y, z) \wedge \alpha(x, y) \leqslant \alpha(x, z)$

for all $x, y, z \in X$.

M. P. Fourman and D. S. Scott. Sheaves and logic. *Lecture Notes in Mathematics*, 753:302–401, 1979.

Ω-sets: Sets valued in a frame

In an Ω-set (X, α), the value

$$\alpha(x, y)$$

is interpreted as the truth-value of the statement

x is equal to y.

Since each entity is supposed to be equal to itself as long as it exists (or, once it is defined), the value

$$\alpha(x, x)$$

may be understood as the extent of existence of x.

Ω-sets: Sets valued in a frame

Example

If $\Omega = \mathbf{2}$, then a $\mathbf{2}$-set (X, α) is just an equivalence relation on a subset of X. Explicitly,

$$\{(x, y) \in X \times X \mid \alpha(x, y) = 1\}$$

is an equivalence relation on the subset

$$\{x \in X \mid \alpha(x, x) = 1\}$$

of X, whose elements are supposed to exist. In particular, (X, α) reduces to a classical set if

- (X, α) is separated, i.e., $\alpha(x, y) = 1$ if and only if $x = y$;
- (X, α) is global, i.e., $\alpha(x, x) = 1$ for all $x \in X$.

Ω-sets: Sets valued in a frame

Example

Given a topological space X, let

$$\mathsf{PC}(X) := \{f \mid f\colon D(f) \longrightarrow \mathbb{R} \text{ is continuous, where } D(f) \subseteq X \text{ is open}\}.$$

For any $f, g \in \mathsf{PC}(X)$, let

$$\alpha(f,g) := \mathrm{Int}\{x \in D(f) \cap D(g) \mid f(x) = g(x)\}.$$

Then $(\mathsf{PC}(X), \alpha)$ is an $\mathcal{O}(X)$-set, where $\mathcal{O}(X)$ is the frame of open subsets of X.

The category of Ω-sets

A morphism $\zeta\colon (X,\alpha) \mathrel{\circ\!\!\longrightarrow} (Y,\beta)$ of Ω-sets is a map

$$\zeta\colon X\times Y\longrightarrow\Omega$$

such that

1. $\beta(y,y')\wedge\zeta(x,y)\wedge\alpha(x',x)\leqslant\zeta(x',y')$,
2. $\zeta(x,y')\wedge\zeta(x,y)\leqslant\beta(y,y')$,
3. $\alpha(x,x)=\displaystyle\bigvee_{y\in Y}\zeta(x,y)$

for all $x,x'\in X$, $y,y'\in Y$. The category of Ω-sets and their morphisms is denoted by

$$\mathbf{\Omega\text{-}Set}.$$

Ω-sets as sheaves on Ω

From an Ω-set (X, α) we may construct a sheaf F on Ω given by

$$Fu = \{x \in X \mid \alpha(x, x) = u\}.$$

Conversely, a sheaf F on Ω gives rise to an Ω-set (X, α), where

$$X = \coprod_{u \in \Omega} Fu \quad \text{and} \quad \alpha(x, y) = \bigvee \{w \leqslant u \wedge v \mid x_w = y_w\}$$

for all $x \in Fu$, $y \in Fv$.

Ω-sets as sheaves on Ω

Theorem

Ω-$\mathbf{Set}$ is equivalent to the category

$$\mathbf{Sh}(\Omega)$$

of sheaves on Ω.

M. P. Fourman and D. S. Scott. Sheaves and logic. *Lecture Notes in Mathematics*, 753:302–401, 1979.

Ω-$\mathbf{Set}$ is a topos

Therefore, Ω-$\mathbf{Set}$ is a topos, because so is $\mathbf{Sh}(\Omega)$.

In this sense, the theory of topoi can be regarded as the categorical foundation for both classical sets and Ω-sets.

What is a quantale-valued set?

For simplicity, let

$$\mathsf{Q} = (\mathsf{Q}, \&, 1)$$

be a commutative and divisible quantale, though we can proceed with an arbitrary quantale Q. Explicitly:

- $(\mathsf{Q}, \&, 1)$ is a commutative monoid;
- Q is a complete lattice;
- $p \,\&\, - : \mathsf{Q} \longrightarrow \mathsf{Q}$ preserves suprema, with the induced right adjoint $\rightarrow$ satisfying

  $$p \,\&\, q \leqslant r \iff p \leqslant q \rightarrow r$$

  for all $p, q, r \in \mathsf{Q}$;
- $u = q \,\&\, (q \rightarrow u)$ whenever $u \leqslant q$ in Q.

What is a quantale-valued set?

Since every frame is a commutative and idempotent quantale, it is natural to ask:

Question

What is a quantale-valued set?

What is a quantale-valued set?

To answer this question, Höhle and his collaborators proposed the notion of

Q-set.

U. Höhle. M-valued sets and sheaves over integral commutative cl-monoids. In *Applications of Category Theory to Fuzzy Subsets*, 33–72, Springer, 1992.

U. Höhle. Presheaves over GL-monoids. In *Non-Classical Logics and their Applications to Fuzzy Subsets*, 127–157, Springer, 1995.

U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

Ω-sets as enriched categories

Every frame Ω induces a quantaloid $\mathsf{D}\Omega$:

- objects of $\mathsf{D}\Omega$ are elements $p, q, r, \dots$ in Ω;
- for $p, q \in \Omega$,
$$\mathsf{D}\Omega(p, q) = \{d \in \Omega \mid d \leqslant p \wedge q\};$$
- for $d \in \mathsf{D}\Omega(p, q)$, $e \in \mathsf{D}\Omega(q, r)$, the composition
$$e \circ d = e \wedge d;$$
- $q \in \mathsf{D}\Omega(q, q)$ is the identity morphism on q.

R. F. C. Walters. Sheaves and Cauchy-complete categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 22(3):283–286, 1981.

Quantaloids

A quantaloid is a category $\mathcal{Q}$ whose hom-sets are complete lattices, such that

$$-\circ u\colon \mathcal{Q}(q,r)\longrightarrow\mathcal{Q}(p,r)\quad\text{and}\quad v\circ -\colon \mathcal{Q}(p,q)\longrightarrow\mathcal{Q}(p,r)$$

preserve suprema, with induced right adjoints

$$(-\circ u)\dashv(-\swarrow u)\colon \mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(q,r),$$
$$(v\circ -)\dashv(v\searrow -)\colon \mathcal{Q}(p,r)\longrightarrow\mathcal{Q}(p,q)$$

satisfying

$$v\circ u\leqslant w\iff v\leqslant w\swarrow u\iff u\leqslant v\searrow w$$

for all $\mathcal{Q}$-arrows $u\in\mathcal{Q}(p,q)$, $v\in\mathcal{Q}(q,r)$ and $w\in\mathcal{Q}(p,r)$.

K. I. Rosenthal. The Theory of Quantaloids. *Longman*, 1996.

$\mathcal{Q}_0$-typed sets

Given a small quantaloid $\mathcal{Q}$, let $\mathcal{Q}_0$ be its set of objects.

A $\mathcal{Q}_0$-typed set is a set X equipped with a type map

$$|\text{-}|\colon X \longrightarrow \mathcal{Q}_0;$$

such that each $x \in X$ has a type $|x| \in \mathcal{Q}_0$.

$\mathcal{Q}$-categories

A category enriched in $\mathcal{Q}$, or $\mathcal{Q}$-category for short, is a $\mathcal{Q}_0$-typed set X equipped with a family of $\mathcal{Q}$-arrows

$$\{\alpha(x,y)\colon |x| \longrightarrow |y|\}_{x,y\in X},$$

such that

(1) $1_{|x|} \leqslant \alpha(x,x)$,

(2) $\alpha(y,z)\circ\alpha(x,y) \leqslant \alpha(x,z)$

for all $x,y,z \in X$.

I. Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory and Applications of Categories*, 14(1):1–45, 2005.

$\mathcal{Q}$-functors

Let (X,α), (Y,β) be $\mathcal{Q}$-categories. A $\mathcal{Q}$-functor

$$f\colon (X,\alpha)\longrightarrow (Y,\beta)$$

is a map $f\colon X\longrightarrow Y$ such that

$$|x|=|fx| \quad \text{and} \quad \alpha(x,x')\leqslant \beta(fx,fx')$$

for all $x,x'\in X$. The category of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors is denoted by

$$\mathcal{Q}\text{-}\mathbf{Cat}.$$

Let (X, α), (Y, β) be $\mathcal{Q}$-categories. A $\mathcal{Q}$-distributor

$$\varphi\colon (X, \alpha) \mathrel{\circ\!\!\!\rightarrow} (Y, \beta)$$

is a map that assigns $\varphi(x, y) \in \mathcal{Q}(|x|, |y|)$ to each pair $(x, y) \in X \times Y$, such that

$$\beta(y, y') \circ \varphi(x, y) \circ \alpha(x', x) \leqslant \varphi(x', y')$$

for all $x, x' \in X$, $y, y' \in Y$.

$\mathcal{Q}$-distributors

The category

$$\color{red}{\mathcal{Q}\text{-}\mathbf{Dist}}$$

of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors is again a (large) quantaloid, with the composition of

$$\varphi\colon (X,\alpha) {\bigcirc\!\!\!\!\!\!\longrightarrow} (Y,\beta) \quad \text{and} \quad \psi\colon (Y,\beta) {\bigcirc\!\!\!\!\!\!\longrightarrow} (Z,\gamma)$$

given by

$$(\psi\circ\varphi)(x,z)=\bigvee_{y\in Y}\psi(y,z)\circ\varphi(x,y),$$

and the identity $\mathcal{Q}$-distributor on (X,α) given by

$$\alpha\colon (X,\alpha) {\bigcirc\!\!\!\!\!\!\longrightarrow} (X,\alpha).$$

$\mathcal{Q}$-distributors

$\mathcal{Q}$-distributors

$$\varphi\colon (X,\alpha) {\circ\!\!\!\longrightarrow} (Y,\beta) \quad \text{and} \quad \psi\colon (Y,\beta) {\circ\!\!\!\longrightarrow} (X,\alpha)$$

form an adjunction in $\mathcal{Q}$-$\mathbf{Dist}$, denoted by

$$\varphi \dashv \psi,$$

if

$$\alpha \leqslant \psi \circ \varphi \quad \text{and} \quad \varphi \circ \psi \leqslant \beta.$$

In this case, φ is called a left adjoint of ψ, and ψ is called a right adjoint of φ.

Ω-sets are symmetric DΩ-categories

A DΩ-category with underlying set X is given by maps

$$|\text{-}|\colon X \longrightarrow \Omega \quad \text{and} \quad \alpha\colon X \times X \longrightarrow \Omega,$$

such that

1. $\alpha(x, y) \leqslant |x| \wedge |y|$,
2. $|x| \leqslant \alpha(x, x)$,
3. $\alpha(y, z) \wedge \alpha(x, y) \leqslant \alpha(x, z)$

for all $x, y, z \in X$.

Ω-sets are symmetric DΩ-categories

Note that the first two inequalities force

$$|x| = \alpha(x, x)$$

for all $x \in X$. Hence, a DΩ-category consists of a set X and a map $\alpha\colon X \times X \longrightarrow \Omega$ such that

1. $\alpha(x, y) \leqslant \alpha(x, x) \wedge \alpha(y, y)$,
2. $\alpha(y, z) \wedge \alpha(x, y) \leqslant \alpha(x, z)$

for all $x, y, z \in X$.

Ω-sets are symmetric DΩ-categories

A DΩ-category (X, α) is symmetric if

$$\alpha(x, y) = \alpha(y, x)$$

for all $x, y \in X$. Therefore:

Theorem

An Ω-set is precisely a symmetric DΩ*-category, and*

$$\Omega\text{-}\mathbf{Set}$$

is the category of symmetric DΩ*-categories and left adjoint* DΩ*-distributors.*

From DΩ to DQ

Every commutative and divisible quantale Q induces a quantaloid DQ:

- objects of DQ are elements $p, q, r, \ldots$ in Q;
- for $p, q \in \mathsf{Q}$,
$$\mathsf{DQ}(p,q) = \{d \in \mathsf{Q} \mid d \leqslant p \wedge q\};$$
- for $d \in \mathsf{DQ}(p,q)$, $e \in \mathsf{DQ}(q,r)$, the composition
$$e \circ d = e \mathbin{\&} (q \to d);$$
- $q \in \mathsf{DQ}(q,q)$ is the identity morphism on q.

U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*, 256:95–116, 2014.

DQ-categories

A DQ-category with underlying set X is given by maps

$$|\text{-}|\colon X \longrightarrow \mathsf{Q} \quad \text{and} \quad \alpha\colon X \times X \longrightarrow \mathsf{Q},$$

such that

1. $\alpha(x,y) \leqslant |x| \wedge |y|$,
2. $|x| \leqslant \alpha(x,x)$,
3. $\alpha(y,z) \mathbin{\&} (|y| \to \alpha(x,y)) \leqslant \alpha(x,z)$

for all $x, y, z \in X$.

DQ-categories

Note that the first two inequalities force

$$|x| = \alpha(x,x)$$

for all $x \in X$. Hence, a DQ-category consists of a set X and a map

$$\alpha \colon X \times X \longrightarrow \mathsf{Q},$$

such that

1. $\alpha(x,y) \leqslant \alpha(x,x) \wedge \alpha(y,y)$,
2. $\alpha(y,z) \mathbin{\&} (\alpha(y,y) \to \alpha(x,y)) \leqslant \alpha(x,z)$

for all $x, y, z \in X$.

Q-sets as symmetric DQ-categories

A DQ-category (X, α) is symmetric if

$$\alpha(x, y) = \alpha(y, x)$$

for all $x, y \in X$.

Definition

A Q-set is a symmetric DQ-category.

U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

Q-sets

Explicitly, a Q-set is a (crisp) set X equipped with a map

$$\alpha\colon X \times X \longrightarrow \mathsf{Q}$$

such that

1. (strictness) $\alpha(x,y) \leqslant \alpha(x,x) \wedge \alpha(y,y)$,
2. (symmetry) $\alpha(x,y) = \alpha(y,x)$,
3. (transitivity) $\alpha(y,z) \mathbin{\&} (\alpha(y,y) \to \alpha(x,y)) \leqslant \alpha(x,z)$

for all $x, y, z \in X$.

U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

In a Q-set (X, α), the value

$$\alpha(x, y)$$

is understood as the truth-value of the statement

x is equal to y.

Since each entity is supposed to be equal to itself as long as it exists (or, once it is defined), the value

$$\alpha(x, x)$$

may be understood as the extent of existence of x.

The axioms of a Q-set are interpreted as follows:

1. (strictness) Each entity is more equal to itself than to any other one.
2. (symmetry) If x is equal to y, then y is equal to x.
3. (transitivity) If y is equal to z, and if the existence of y forces x to be equal to y, then x is equal to z.

H. Lai, L. Shen, Y. Tao and D. Zhang. Quantale-valued dissimilarity. *Fuzzy Sets and Systems*, 390:48–73, 2020.

Example

Let Q be the Lawvere quantale

$$[0,\infty] = ([0,\infty]^{\mathrm{op}}, +, 0).$$

Then $[0,\infty]$-sets are symmetric partial metric spaces; that is, sets X equipped with a map

$$\alpha\colon X \times X \longrightarrow [0,\infty]$$

such that

1. $\alpha(x,x) \vee \alpha(y,y) \leqslant \alpha(x,y)$,
2. $\alpha(x,y) = \alpha(y,x)$,
3. $\alpha(x,z) \leqslant \alpha(y,z) - \alpha(y,y) + \alpha(x,y)$

for all $x,y,z \in X$.

Q-sets

In particular, let

$$\mathcal{I} := \{[a, b] \mid 0 \leqslant a < b \leqslant \infty\}$$

be the set of closed intervals contained in $[0, \infty]$. Then

$$\alpha([a, b], [c, d]) = (b \vee d) - (a \wedge c)$$

defines a symmetric partial metric space $(\mathcal{I}, \alpha)$.

F. W. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matématico e Fisico di Milano*, XLIII:135-166, 1973.

S. G. Matthews. Partial metric topology. *Annals of the New York Academy of Sciences*, 728(1):183-197, 1994.

Let

$$\mathcal{Q}\text{-}\mathbf{Set}$$

denote the category of symmetric D$\mathcal{Q}$-categories and left adjoint D$\mathcal{Q}$-distributors.

It is natural to ask:

Question

Is $\mathcal{Q}$-$\mathbf{Set}$ a topos?

Q-$\mathbf{Set}$ is not generally a topos

The following theorem was proposed as a conjecture by Pu–Zhang, and proved in our recent work:

Theorem

For a commutative and divisible quantale Q, *the category* Q-$\mathbf{Set}$ *is a topos if, and only if,* Q *is a frame.*

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

X. Hu and L. Shen. Q-$\mathbf{Set}$ is not generally a topos. *Fuzzy Sets and Systems*, 517:109484, 2025.

Q-$\mathbf{Set}$ is not generally a topos

Therefore, the theory of topoi is unlikely to serve as the categorical foundation of Q-sets when the quantale Q is non-idempotent.

Sketch of the proof

Proposition

The category $\mathcal{Q}$-$\mathbf{Set}$ *is equivalent to the full subcategory*

$$\mathcal{DQ}\text{-}\mathbf{CcSymCat}$$

of $\mathcal{DQ}$-$\mathbf{Cat}$ *whose objects are separated and Cauchy complete* $\mathcal{Q}$*-sets.*

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

Sketch of the proof

Proposition

Every monic arrow in a topos is an equalizer.

Proposition

Let $\mathcal{E}$ be a topos. For each $X \in \operatorname{ob}\mathcal{E}$, the partially ordered set $\operatorname{Sub} X$ is a lattice, in which the intersection $U \cap V$ of subobjects U, V of X is given by the pullback on the left below, and their union $U \cup V$ is given by the pushout on the right below.

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d, tail] \\ U \arrow[r, tail] & X \end{tikzcd} \qquad \begin{tikzcd} U \cap V \arrow[r, tail] \arrow[d, tail] & U \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd}$$

S. Mac Lane and I. Moerdijk. Sheaves in Geometry and Logic: A First Introduction to Topos Theory. *Springer*, 1992.

Sketch of the proof

For each $p, q \in \mathsf{Q}$, define

$$C_q := \{p \leqslant q \mid p \,\&\, (q \to p) = p\}$$

and

$$p \sqsubseteq q \iff p \in C_q.$$

Then $\sqsubseteq$ is a partial order on Q.

Sketch of the proof

For each $q \in \mathsf{Q}$, note that the Cauchy completion of the one-element Q-set $\{q\}$ is precisely

$$(C_q, \wedge).$$

Lemma

Suppose that DQ-$\mathbf{CcSymCat}$ *is a topos. Then for each* $q \in \mathsf{Q}$*, every subobject of* $(C_q, \wedge)$ *in* DQ-$\mathbf{CcSymCat}$ *is of the form*

$$(C_p, \wedge)$$

for some $p \sqsubseteq q$*.*

Sketch of the proof

Lemma

Suppose that $\mathsf{D}\mathsf{Q}\text{-}\mathbf{CcSymCat}$ *is a topos. Then*

$$p \sqcap q = p \wedge q$$

for all $p, q \in \mathsf{Q}$.

Sketch of the proof

This lemma is proved by exploring intersections and unions in the lattice

$$\mathrm{Sub}(\mathsf{Q}, \wedge)$$

of subobjects of $(\mathsf{Q}, \wedge)$ in DQ-$\mathbf{CcSymCat}$:

- The intersection of $(C_p, \wedge)$ and $(C_q, \wedge)$ in $\mathrm{Sub}(\mathsf{Q}, \wedge)$ is

$$(C_{p \sqcap q}, \wedge).$$

- The union of $(C_p, \wedge)$ and $(C_q, \wedge)$ in $\mathrm{Sub}(\mathsf{Q}, \wedge)$ is given by the Cauchy completion of the Q-set (Z, γ) with

$$Z = \{p, q\}$$

and

$$\gamma(p, p) = p, \quad \gamma(q, q) = q, \quad \gamma(p, q) = \gamma(q, p) = p \sqcap q.$$

Sketch of the proof

Therefore, in the case that DQ-$\mathbf{CcSymCat}$ is a topos, we have

$$q \sqcap 1 = q \wedge 1 = q$$

for all $q \in \mathsf{Q}$, and consequently

$$q \sqsubseteq 1,$$

which means that

$$q \mathbin{\&} q = q \mathbin{\&} (1 \rightarrow q) = q.$$

Hence, every element of Q is idempotent, showing that Q is a frame.

Cartesian closedness of $[0,1]_*$-$\mathbf{Set}$

Consider the quantale

$$[0,1]_* = ([0,1], *, 1),$$

where $*$ is a continuous t-norm on the unit interval $[0,1]$.

Corollary

Let $*$ be a continuous t-norm on the unit interval $[0,1]$. Then the category of $[0,1]_*$-sets is a topos if, and only if, $*$ is the minimum t-norm on $[0,1]$.

X. Hu and L. Shen. $\mathcal{Q}$-$\mathbf{Set}$ is not generally a topos. *Fuzzy Sets and Systems*, 517:109484, 2025.

Cartesian closedness of $[0,1]_*$-$\mathbf{Set}$

Theorem

Let $*$ be a continuous t-norm on the unit interval $[0,1]$. Then the category of $[0,1]_*$-sets is cartesian closed if, and only if, $*$ is the minimum t-norm on $[0,1]$.

L. Shen and J. Zhang. Cartesian closedness of the category of real-valued sets, I. *Fuzzy Sets and Systems*, 531:109767, 2026.

The full subcategory of D$[0,1]_*$-$\mathbf{Cat}$ consisting of separated Cauchy complete $[0,1]_*$-sets is denoted by

$$[0,1]_*\text{-}\underline{\mathbf{CcSet}}.$$

Proposition

The category $[0,1]_*$-$\mathbf{Set}$ *is equivalent to* $[0,1]_*$-$\underline{\mathbf{CcSet}}$.

Q. Pu and D. Zhang. Preordered sets valued in a GL-monoid. *Fuzzy Sets and Systems*, 187(1):1–32, 2012.

Sketch of the proof

Lemma

For $p, q \in [0,1]$, the following statements are equivalent:

1. *There exists a morphism*

$$\varphi\colon \{p\} \mathrel{-\!\!\circ\!\!\rightarrow} \{q\}$$

   of one-element $[0,1]_*$-sets.

2. *Either $p = q$, or else $p < q$ and p is idempotent.*

In this case, it necessarily holds that $\varphi(p,q) = p$. Therefore, a morphism between one-element $[0,1]_*$-sets, when it exists, will be denoted by

$$\overline{p}\colon \{p\} \mathrel{-\!\!\circ\!\!\rightarrow} \{q\}.$$

Sketch of the proof

For $Y, Z \in [0,1]_*$-$\underline{\mathbf{CcSet}}$, if the exponential Z^Y exists in $[0,1]_*$-$\underline{\mathbf{CcSet}}$, then there are bijections

$$\begin{aligned} Z_q^Y &\cong [0,1]_*\text{-}\mathbf{Set}(\{q\}, Z^Y) \\ &\cong [0,1]_*\text{-}\underline{\mathbf{CcSet}}(\mathsf{C}^\dagger\{q\}, Z^Y) \\ &\cong [0,1]_*\text{-}\underline{\mathbf{CcSet}}(\mathsf{C}^\dagger\{q\} \times Y, Z) \end{aligned}$$

natural in q. We denote the above bijections by

$$\mathsf{Y}, \ \mathsf{R} \text{ and } \mathsf{M},$$

respectively. Thus, every $f \in Z_q^Y$ is mapped to

$$\mathsf{MRY}f \in [0,1]_*\text{-}\underline{\mathbf{CcSet}}(\mathsf{C}^\dagger\{q\} \times Y, Z)$$

under these bijections.

Sketch of the proof

For $\mathsf{D}[0,1]_*$-functors

$$f\colon \mathsf{C}^\dagger\{p\} \times Y \longrightarrow Z \quad \text{and} \quad g\colon \mathsf{C}^\dagger\{q\} \times Y \longrightarrow Z,$$

we define a subset $D(f,g)$ of the interval $[0,1]$, which consists of $u \in [0, p \wedge q]$ satisfying

$$\big(u * (p \to r) * (q \to r')\big) \wedge 1_Y^\natural(y,y') \leqslant 1_Z^\natural\left(f(\overline{r},y), g(\overline{r'},y')\right)$$

for all $(\overline{r},y) \in \mathsf{C}^\dagger\{p\} \times Y$ and $(\overline{r'},y') \in \mathsf{C}^\dagger\{q\} \times Y$.

Sketch of the proof

Lemma

For $Y, Z \in [0,1]_*$-$\underline{\mathbf{CcSet}}$, suppose that the exponential Z^Y exists in $[0,1]_*$-$\underline{\mathbf{CcSet}}$. Then

$$1^\natural_{Z^Y}(f,f) = p$$

if the domain of the $\mathsf{D}[0,1]_*$*-functor* $\mathsf{MRY}f$ *is* $\mathsf{C}^\dagger\{p\} \times Y$*, and*

$$1^\natural_{Z^Y}(f,g) = \bigvee D(\mathsf{MRY}f, \mathsf{MRY}g) \tag{1}$$

for all $f, g \in Z^Y$*.*

Sketch of the proof

Suppose that $*$ is not the minimum t-norm on $[0, 1]$. Then there exists a non-trivial closed interval

$$[a, b] \subseteq [0, 1]$$

such that the continuous t-norm $[a, b]_*$ obtained by restricting $*$ to $[a, b]$ is either isomorphic to the product t-norm $[0, 1]_\times$ or isomorphic to the Łukasiewicz t-norm $[0, 1]_*$.

Let us consider the $[0, 1]_*$-set X with

$$X = \{x, x'\}, \quad 1_X^\natural(x, x) = 1_X^\natural(x', x') = b \quad \text{and} \quad 1_X^\natural(x, x') = a.$$

Let $Y = \mathsf{C}^\dagger\{b\}$ and $Z = \mathsf{C}^\dagger X$. Then it can be shown that Z^Y equipped with (1) cannot be a $[0, 1]_*$-set. Therefore, $[0, 1]_*$-$\underline{\mathbf{CcSet}}$ cannot be cartesian closed.

Future works

- Investigate the cartesian closedness of Q-$\mathbf{Set}$ for a general quantale Q.
- Search for a categorical framework for Q-sets beyond topoi.

Thank you!