

Homodular pseudofunctors as objective invariants

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Pacific Category Theory Seminar

How it started and has not yet ended

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- ▶ The first technique is to construct a universal homological functor and prove the retraction property for it so that the property holds automatically for any homology.
- ▶ The second idea is to use the calculus of (virtual) Joyal species where differentiation involves a shift $n \mapsto n + 1$ as in the derivative of a Taylor series. Integration gives the retraction.

Acknowledgements

I was visiting SUNY Buffalo in 1985. On 21 June 1985, Bill Lawvere gave a seminar on Joyal's work. Presumably he had a preprint of the *Comptes Rendues* article which appeared later in the year.

This year Nathanael Arkor provided very helpful feedback on the first arXiv preprint on homodular pseudofunctors, much like a referee's report, by directing me to many references and by suggesting further examples. The revised development of the first technique developed because of Nathanael's prodding to cover those examples.

My current versions of Joyal's two techniques have yet to conflow into an application to the $GL_n(\mathbb{F}_q)$ sequence.

First technique: Universal “homological” functors

Let gpd be the category of finite groupoids, gpd_{df} the subcategory with morphisms restricted to discrete fibration, and \mathcal{A} an additive category. A *homological functor* (according to Joyal in 1985) consists of functors $H : \text{gpd} \rightarrow \mathcal{A}$ and $\Sigma : \text{gpd}_{\text{df}}^{\text{op}} \rightarrow \mathbf{HB}$ such that

- ▶ naturally isomorphic functors are taken to equal morphisms by H ;
- ▶ H, Σ obey the Chevalley-Beck-type condition on pullbacks of discrete fibrations along arbitrary functors between finite groupoids;
- ▶ H preserves finite coproducts.

The universal homological functor is obtained via an abstraction of a 2-category which uses finite coproduct completion (that is, fam).

Fibrations in a bicategory

Suppose the bicategory \mathcal{K} admits slices (“comma objects”). Each morphism $E \xrightarrow{p} A$ in \mathcal{K} induces a morphism $E \downarrow E \xrightarrow{\bar{p}} A \downarrow p$ and isomorphisms $ps \cong s\bar{p}$ and $t\bar{p} \cong t$ which paste onto the 2-cell $\lambda_{A,p}$ to yield $p\lambda_{E,E}$. The morphism $E \xrightarrow{p} A$ will be called a *fibration* (although it should properly be called a “pseudofibration”) when \bar{p} has a right adjoint with invertible counit in the bicategory \mathcal{K} .

$$\begin{array}{ccccc}
 E \downarrow E & \xrightarrow{\bar{p}} & A \downarrow E & \xrightarrow{t} & E \\
 \downarrow s & \xRightarrow{\cong} & \downarrow s & \xRightarrow{\lambda_{A,p}} & \downarrow p \\
 E & \xrightarrow{p} & A & \xrightarrow{1_A} & A
 \end{array}$$

Fibrations have the property that pullbacks along them are automatically pseudopullbacks [42] and hence bipullbacks in Cat .

Cofibrations in a bicategory

An *opfibration* is a fibration in \mathcal{K}^{co} . A *cofibration* is a fibration in \mathcal{K}^{op} .

A *coopfibration* is a fibration in $\mathcal{K}^{\text{coop}}$. A full sub- \mathcal{V} -category A of a \mathcal{V} -category X is called a *sieve* ("crible" in French) when, for all $x \in X$, if $x \notin A$ then $X(x, a) = 0$ for all $a \in A$.

If \mathcal{K} admits coslices, a morphism $C \xrightarrow{u} A$ is a cofibration if and only if the canonical morphism $C \uparrow u \xrightarrow{\bar{u}} A \uparrow A$ induced by \bar{u} has a left adjoint ℓ with invertible counit $\ell \circ \bar{u} \cong 1_{C \uparrow u}$.

$$\begin{array}{ccccc}
 A \uparrow A & \xrightarrow{\ell} & C \uparrow u & \xrightarrow{\ell \circ \bar{u}} & C \uparrow u \\
 & \perp & & \Downarrow \cong & \\
 & \xleftarrow{\bar{u}} & & \xrightarrow{1_{C \uparrow u}} & \\
 & & & &
 \end{array}$$

Proposition

Every cofibration $A \rightarrow X$ in $\mathcal{V}\text{-Cat}$ is equivalent under A to the inclusion of a sieve. In particular, every cofibration in $\mathcal{V}\text{-Cat}$ is a fully faithful \mathcal{V} -functor.

Homodular pseudofunctors

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ j \downarrow & \Rightarrow & \downarrow h \\ Y & \xrightarrow{k} & Q \end{array}$$

Definition

A pseudofunctor $T : \mathcal{K} \rightarrow \mathcal{X}$ between bicategories is *homodular* when

- H1. for all cofibrations i in \mathcal{K} , the morphism Ti has a right adjoint Ti^* ;
- H2. for all bipushout squares in \mathcal{K} (as above) with i a cofibration and j a coopfibration, the mate $Tj \circ Ti^* \Rightarrow Tk^* \circ Th$, of the image under T of the isomorphism in the square, is invertible.

Proposition

Suppose $T : \mathcal{K} \rightarrow \mathcal{X}$ is a homodular pseudofunctor where \mathcal{K} admits coslices. Then, for all $B \xrightarrow{f} A$ in \mathcal{K} , the morphism Tf has a right adjoint Tf^ .*

Proposition

Suppose $T : \mathcal{K} \rightarrow \mathcal{X}$ is a homodular pseudofunctor where \mathcal{K} admits coslices. Then, for all $B \xrightarrow{f} A$ in \mathcal{K} , the morphism Tf has a right adjoint Tf^* .

Proof.

In the coslice diagram

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ f \downarrow & \xleftarrow{\gamma} & \downarrow i \\ A & \xrightarrow{j} & B \uparrow f \end{array},$$

the morphism i is a cofibration while j has a left adjoint j' with $j' \circ j \cong 1_A$ and $j' \circ i \cong f$. So $Tf \cong Tj' \circ Ti$ has right adjoint $Ti^* \circ Tj$ using H1. \square

New homomodularity from old

A pseudofunctor $\mathcal{X} \xrightarrow{F} \mathcal{H}$ is *fully faithful* when each of its effects $\mathcal{X}(X, Y) \xrightarrow{F} \mathcal{H}(FX, FY)$ on hom categories is an equivalence.

Proposition

Suppose $F : \mathcal{X} \rightarrow \mathcal{H}$ is a fully faithful pseudofunctor. The pseudofunctor $T : \mathcal{K} \rightarrow \mathcal{X}$ is homomodular if and only if the composite $F \circ T : \mathcal{K} \rightarrow \mathcal{H}$ is homomodular.

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Proposition

Suppose \mathcal{H} has bipushouts and coslices preserved by $S : \mathcal{H} \rightarrow \mathcal{K}$. If $T : \mathcal{K} \rightarrow \mathcal{X}$ is homodular then so is $T \circ S : \mathcal{H} \rightarrow \mathcal{X}$.

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Proposition

If $T : \mathcal{H} \rightarrow \mathcal{X}$ is homodular then, for any bicategory \mathcal{A} , so is

$$T \circ - : \text{Hom}(\mathcal{A}, \mathcal{H}) \rightarrow \text{Hom}_{\text{Int}}(\mathcal{A}, \mathcal{X}).$$

Examples

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- ▶ Suppose X is a complete category. Then $[-, X] : \text{Cat}^{\text{op}} \rightarrow \text{CAT}$ is homomodal.
- ▶ The pseudofunctor $(-)^* : \text{Cat}^{\text{op}} \rightarrow \text{Mod}^{\text{co}}$ is homomodal.
- ▶ (Rosebrugh-Wood) Let TOP denote the bicategory of topoi and geometric morphisms. Let TLEX be the bicategory of topoi and finite-limit-preserving (= left-exact) functors. The pseudofunctor $(-)_* : \text{TOP} \rightarrow \text{TLEX}^{\text{co}}$, taking each geometric morphism to its left-exact left adjoint, is homomodal.

Universal homomodularity

A homodular pseudofunctor $\mathcal{K} \xrightarrow{S} \mathcal{M}$ is *universal* when, for each homodular pseudofunctor $\mathcal{K} \xrightarrow{T} \mathcal{X}$, there is a pseudofunctor $\bar{T} : \mathcal{M} \rightarrow \mathcal{X}$, unique up to equivalence, with the property $\bar{T} \circ S \simeq T$. A pseudofunctor $E : \mathcal{K} \rightarrow \mathcal{M}$ is *essentially surjective on objects* when, for each object $X \in \mathcal{M}$, there exists an object $K \in \mathcal{K}$ with $EK \simeq X$.

Proposition

Universal homodular pseudofunctors are essentially surjective on objects.

Proof.

Factor the universal $\mathcal{K} \xrightarrow{S} \mathcal{D}$ as $S \simeq F \circ E$ where $\mathcal{K} \xrightarrow{E} \mathcal{M}$ is the identity on objects and $\mathcal{M} \xrightarrow{F} \mathcal{D}$ is fully faithful. Also, E is homodular. S universal gives a pseudofunctor $\mathcal{D} \xrightarrow{R} \mathcal{M}$ with $R \circ S \simeq E$. Then $F \circ R \circ S \simeq 1_{\mathcal{D}} \circ S$ and by uniqueness, $F \circ R \simeq 1_{\mathcal{D}}$. By uniqueness of the factorization, $R \circ F \circ E \simeq E$, $F \circ R \circ F \simeq F$ imply $R \circ F \simeq 1_{\mathcal{M}}$. So F is a biequivalence and $S \simeq F \circ E$ is essentially surjective on objects. \square

Finitary cosmoi

Recall from Section 6 of Carboni-Johnson-St-Verity [46: Modulated bicategories] that an *finitary cosmos* is a bicategory \mathcal{M} possessing

- (a) finite bicategorical coproducts;
- (b) a Kleisli object for every monad;
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As mentioned atop p.266 of [46], the construction proving Proposition 1 of [18 = 85: Cauchy characterisation of enriched categories] shows that any finitary cosmos admits finite collages.

Write \mathcal{M}_* for the subcategory of \mathcal{M} with morphisms restricted to the f with right adjoints f^* . Write $\mathcal{I} : \mathcal{M}_* \rightarrow \mathcal{M}$ for the inclusion pseudofunctor.

Module equipment

This is basically Wood's notion of *proarrow equipment*. Indeed, as pointed out by Nathanael Arkor, the axioms of Rosebrugh-Wood [Proarrows and cofibrations (JPAA 1988)] imply that \mathcal{M} is a finitary cosmos.

Definition

Module equipment for a bicategory \mathcal{K} consists of a finitary cosmos \mathcal{M} and a pseudofunctor $Q : \mathcal{K} \rightarrow \mathcal{M}_*$ with the following properties:

- (i) Q is essentially surjective on objects,
- (ii) Q is locally fully faithful (meaning, fully faithful on homcategories),
- (iii) a morphism out of any finite collage in \mathcal{M} is in the essential image of $\mathcal{J} := (\mathcal{K} \xrightarrow{\mathcal{I} \circ Q} \mathcal{M})$ if and only if its composites with all the injections in the lax cocone are in the essential image of \mathcal{J} .

Helpful facts

Proposition

If a bicategory \mathcal{K} admits module equipment \mathcal{J} then coslices exist in \mathcal{K} and are preserved by \mathcal{J} .

Corollary

Module equipments preserve cofibrations.

Proposition

Suppose $X \xleftarrow{i} A \xrightarrow{j} Y$ is a span in \mathcal{K} with module equipment $\mathcal{Q} : \mathcal{K} \rightarrow \mathcal{M}_$. Suppose in \mathcal{M} the unit of $i \dashv i^*$ is invertible. Then a bipushout of the span exists in \mathcal{K} and is preserved by the pseudofunctor $\mathcal{J} : \mathcal{K} \rightarrow \mathcal{M}$. Moreover, in \mathcal{M} , $k \dashv k^*$ has invertible unit and the mate $j \circ i^* \Rightarrow k^* \circ h$ of $k \circ j \cong h \circ i$ is invertible.*

Universality theorem

Theorem

The pseudofunctor $\mathcal{J} : \mathcal{K} \rightarrow \mathcal{M}$ of module equipment for \mathcal{K} is universal homomodal out of \mathcal{K} .

Proof.

Clearly H1 is satisfied since \mathcal{J} factors through \mathcal{M}_* and pseudofunctors preserve adjunctions. Proposition 11 gives H2.

Now take a homomodal pseudofunctor $T : \mathcal{K} \rightarrow \mathcal{X}$. For each morphism $B \xrightarrow{m} A$ in \mathcal{M} , define $\bar{T}m = (Ti_m)^* \circ Tj_m$. This definition is forced since we have $m \cong i_m^* \circ j_m$ in \mathcal{M} and \bar{T} is to be a pseudofunctor essentially restricting to T .

We must see that \bar{T} coherently preserves composition up to isomorphism. So consider composable morphisms $C \xrightarrow{n} B \xrightarrow{m} A$ and see the arXiv for more details. □

Example

If \mathcal{M} is a finitary cosmos, the inclusion pseudofunctor $\mathcal{I} : \mathcal{M}_* \rightarrow \mathcal{M}$ is universal homodular. This is the case where the \mathcal{Q} of Theorem 12 is the identity of $\mathcal{H} = \mathcal{M}_*$.

Example

The universal homodular pseudofunctor with domain $\mathcal{V}\text{-Cat}$ is $(-)_* : \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Mod}$.

Corollary

For any homodular pseudofunctor $T : \mathcal{V}\text{-Cat} \rightarrow \mathcal{X}$ and coslice of the span $A \xleftarrow{u} C \xrightarrow{v} B$ in $\mathcal{V}\text{-Cat}$, the mate $Tu \circ Tv^ \implies Ti^* \circ Tj$ of $T\gamma$ is invertible.*

Monoidal and coproduct structures

Proposition

Suppose \mathcal{X} is a monoidal bicategory and $T : \mathcal{V}\text{-Cat} \rightarrow \mathcal{X}$ is a homodular pseudofunctor with extension $\bar{T} : \mathcal{V}\text{-Mod} \rightarrow \mathcal{X}$. If T is strong monoidal then so is \bar{T} such that the canonical pseudonatural equivalence $\bar{T} \circ (-)_ \simeq T$ is monoidal.*

Proposition

Suppose $T : \mathcal{V}\text{-Cat} \rightarrow \mathcal{X}$ is a homodular pseudofunctor with extension $\bar{T} : \mathcal{V}\text{-Mod} \rightarrow \mathcal{X}$. If T preserves coproducts then \bar{T} preserves coproducts and preserves coproducts in the homcategories.

Second technique: Calculus of species

1. The number of one-dim subspaces of \mathbb{F}_q^n is the n -th q -integer

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + q^3 + \dots + q^{n-1} .$$

2. The number of complete flags $0 < V_1 < V_2 < \dots < V_n = V$ in an n -dimensional \mathbb{F}_q -vector space V is the q -factorial

$$[n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q .$$

3. If V is an n -dim \mathbb{F}_q -vector space, the number of linear bijections from V to itself is

$$\#\mathrm{GL}(n, q) = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1}) = q^{\frac{1}{2}n(n-1)} \phi_n(q) ,$$

$$\text{where } \phi_n(q) = (q^n - 1)(q^{n-1} - 1) \dots (q - 1) = (q - 1)^n [n]_q! .$$

4. The number of k -dim subspaces of V is the q -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{\phi_n(q)}{\phi_k(q) \phi_{n-k}(q)} .$$

The q -Derivative

1. The q -derivative of a function $f(x)$ is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{qx - x}.$$

2. A q -integral [F.H. Jackson (1910)] of a function $f(x)$ is defined by

$$(I_q f)(x) = \sum_{k \geq 0} (1 - q)xq^k f(q^k x).$$

3. We have the q -Leibniz formula:

$$\begin{aligned} D_q(f \cdot g)(x) &= (D_q f)(x)g(qx) + f(x)(D_q g)(x) \\ &= (D_q f)(x)g(x) + f(qx)(D_q g)(x). \end{aligned}$$

Reps of the q -general linear gpd \mathcal{GL}_q as per Joyal-St [49]

The external tensor product $M \otimes N$ of $[\mathcal{GL}_q, \text{Vect}_{\mathbb{C}}]$ is defined by

$$(M \otimes N)(A) = \bigoplus_{C \leq A} M(C) \otimes N(A/C) .$$

The *dimension series* of an $M : \mathcal{GL}_q \rightarrow \text{vect}_{\mathbb{C}}$ with finite dim values is

$$(\dim M)(x) = \sum_{n \geq 0} \dim M(\mathbb{F}_q^n) \frac{x^n}{\phi_n(q)}$$

Using the q -binomial coefficients we can see that

$$\dim(M \otimes N) = \dim M \cdot \dim N .$$

Now we would like to modify this a little and define a q -Taylor dimension series which also takes tensor product to product.

$$(\text{dit } M)(x) = \sum_{n \geq 0} \dim M(\mathbb{F}_q^n) \frac{x^n}{[n]_q!} = (\dim M)((q-1)x)$$

$$\text{dit}(M \otimes N) = \text{dit } M \cdot \text{dit } N .$$

Objective q -derivative

Proposition

The q -derivative of

$$f(x) = \sum_{n \geq 0} a_n \frac{x^n}{[n]_q!}$$

is

$$(D_q f)(x) = \sum_{n \geq 0} a_{n+1} \frac{x^n}{[n]_q!} .$$

So we define the derivative functor $M' : \mathcal{GL}_q \rightarrow \text{Vect}_{\mathbb{C}}$ of M by

$$M'(A) = M(A \oplus \mathbb{F}_q)$$

to guarantee

$$\text{dit} M' = D_q \text{dit} M .$$

Grassmannian q -species

The k -Grassmannian q -species is the functor $\text{Gr}_k : \mathcal{GL}_q \rightarrow \text{Set}$ whose value at $A \in \mathcal{GL}_q$ is the set $\text{Gr}_k(A) = \begin{bmatrix} A \\ k \end{bmatrix}_q$ of k -dimensional subspaces of A ; [49; p. 912]. So

$$\#\text{Gr}_k(\mathbb{F}_q^n) = \begin{bmatrix} n \\ k \end{bmatrix}_q ,$$

With some work (using reduction to row echelon form) we have a natural isomorphism

$$\text{Gr}'_k \cong \text{Gr}_k \times \mathbb{F}_q^k + \text{Gr}_{k-1}$$

for the derivative of the k -Grassmannian which is an objective form of the easily proved abstract equation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q .$$

The objective q -Leibniz rule

Each functor $M : \mathcal{GL}_q \rightarrow \text{Vect}_{\mathbb{C}}$ determines a functor $M^{\$} : \mathcal{GL}_q \rightarrow \text{Vect}_{\mathbb{C}}$ defined by $M^{\$}(A) = A \cdot M(A)$ (the direct sum of $\#A$ copies of $M(A)$).

Using the formula for the derivative of the Grassmannian, we can prove:

Theorem

$$(M \otimes N)' \cong M^{\$} \otimes N' \oplus M' \otimes N$$

My Question: *Have any of the differential category experts out there covered this type of variant to the Leibniz property?*

Thank you one and all!

