A direct-categorical approach to opetopic sets and opetopes

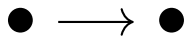
Taichi Uemura (Nagoya University)

January 16, 2026
Pacific Category Theory Seminar

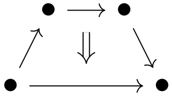
Opetopes

Geometric shapes of many-in-single-out operators in higher dimension.
Used for defining weak ω-categories.

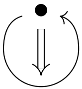
0-opetope



1-opetope



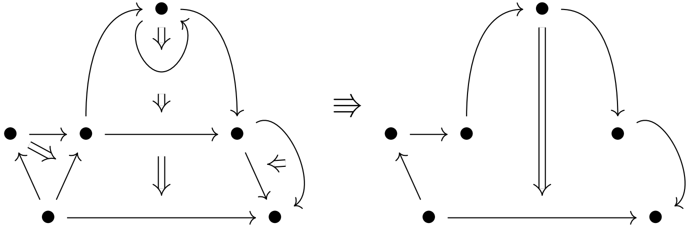
2-opetope with three sources



2-opetope with no source

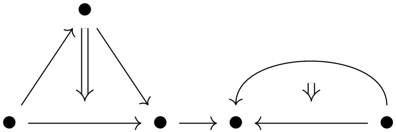
Opetopes

A 3-opetope.



Opetopic sets

The opetopes form a category $\mathbb{O}$. An opetopic set is a set-valued presheaf on $\mathbb{O}$, i.e. a formal colimit of opetopes.



Formal definitions

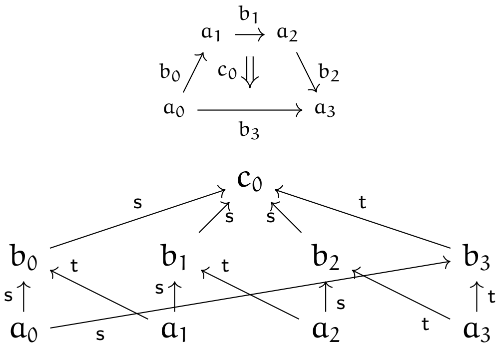
Contributors	Prerequisites
Baez and Dolan (1998)	operad
Leinster (2004)	cartesian monad
Hermida, Makkai, and Power (2002)	multicategory
Kock, Joyal, Batanin, and Mascari (2010)	polynomial monad
Curien, Ho Thanh, and Mimram (2022)	type theory

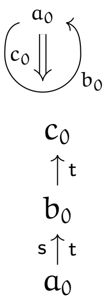
Not sufficiently accessible: some amount of prerequisites; too long.

Posetal approach

Leclerc (2024) proposes a posetal definition of opetopes.

- An opetope is a poset of cells ordered by the subcell relation.
- Subcells of codimension 1 are source or target or both.

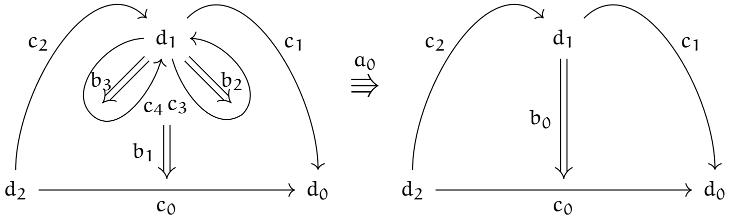




Elementary, simple, and elegant, except the following issue.

Loop issue in posetal approaches

There is no way to distinguish loops, since swapping adjacent loops does not cahnge the subcell relation.



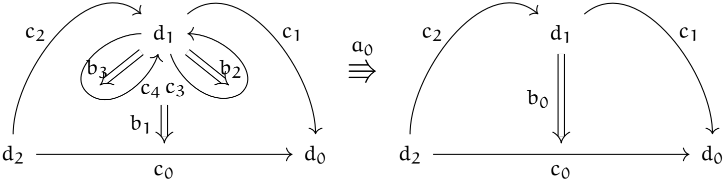
Total ordering on loops is part of structure in Leclerc's definition.

Contribution

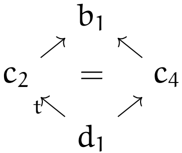
I propose new definitions of opetopes and opetopic sets.

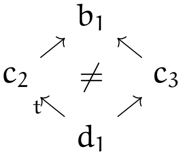
- Take a categorical approach rather than a posetal one.
- No need for special treatment of loops.
- The category of opetopic sets is defined first.
- Opetopes are opetopic sets satisfying one more axiom.
- The only prerequisite is basic category theory.
- Less than two pages in A4 size.
- Equivalent to existing definitions.

Loop issue resolved



Loops (c_3 and c_4) can be distinguished by equality of arrows.





Outline

Outline

Foundations

We work in Univalent Foundations. Constructively fine: no excluded middle; no choice axiom; no propositional resizing.
Non-univalent audience may interpret types as groupoids (Hofmann and Streicher 1998) for this talk.

Gaunt categories

Definition

A category is **gaunt** if its type of objects is a set.

In non-univalent foundations, a category is gaunt if the identities are the only isomorphisms in it (Barwick and Schommer-Pries 2021).

Example

The poset ω of finite ordinals is a gaunt category (so is any poset).

ω-direct categories

Definition

An ω-**direct category** is a gaunt category A equipped with a conservative functor $\mathbf{deg} : A \to \omega$ called the **degree functor**. A k-**step arrow**, written $f : x \to^k y$, is an arrow such that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$. Let $\mathbf{Arr}^k(A)$ denote the set of k-step arrows. Let $A \downarrow^k x \subset A \downarrow x$ denote the subset spanned by k-step arrows into x.

Preopetopic sets

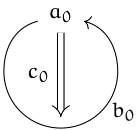
Definition

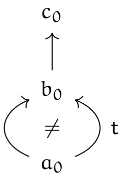
A **preopetopic set** is an ω-direct category A equipped with a subset $\mathbf{S}(A) \subset \mathbf{Arr}^1(A)$ with complement $\mathsf{T}(A)$. A **source arrow**, written $f : x \to^{s} y$, is an arrow in $\mathbf{S}(A)$. A **target arrow**, written $f : x \to^{t} y$, is an arrow in $\mathsf{T}(A)$.

We think of objects in a preopetopic set A as **cells**, and the arrows in A determine the configuration of the cells.

Loops

Opetopes without loop will be encoded as posets.
For a loop, the source and target inclusions will be distinct.





Opetopic set axioms

An **opetopic set** is a preopetopic set A satisfying eight axioms.

Axiom (O1)

$A \downarrow^1 x$ is finite for every $x : A$.

Each cell has finitely many sources and targets.

Definition

A set A is **finite** if there exist $n : \mathbb{N}$ and $e : \{x : \mathbb{N} \mid x < n\} \simeq A$.

Opetopic set axioms

Axiom (O2)

For every object $x : A$ of degree ≥ 1, there exists a unique target arrow into x.

This expresses the single-out nature of opetopes.

Axiom (O3)

For every object $x : A$ of degree 1, there exists a unique source arrow into x.

This expresses that the 1-opetope $(\bullet \to \bullet)$ is single-in.

Homogeneous/heterogeneous factorizations

Definition

Let A be a preopetopic set, $f : y \to^1 x$, and $g : z \to^1 y$. We say (f, g) is **homogeneous** if either

- both f and g are source arrows; or
- both f and g are target arrows.

We say (f, g) is **heterogeneous** if either

- f is a source arrow and g is a target arrow; or
- f is a target arrow and g is a source arrow.

By a **homogeneous/heterogeneous factorization** of a 2-step arrow h we mean a factorization $h = f \circ g$ such that (f, g) is homogeneous/heterogeneous.

Opetopic set axioms

Axiom (O4)

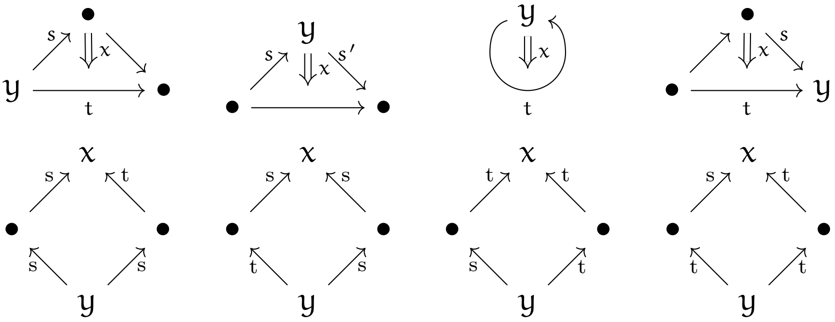
Every 2-step arrow in A has a unique homogeneous factorization.

Axiom (O5)

Every 2-step arrow in A has a unique heterogeneous factorization.

Two factorizations

For example, a 0-cell y is embedded into a 2-cell x in exactly two ways, one is homogeneous and the other is heterogeneous.



Cf. “diamond property” (McMullen and Schulte 2002), “oriented thinness” (Hadzihasanovic 2020).

Opetopic set axioms

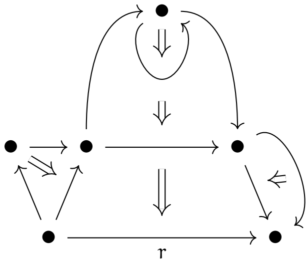
Axiom (O6)

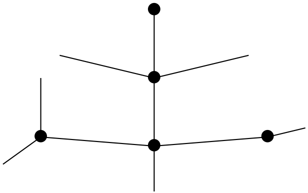
For every object $x : A$ of degree ≥ 2, there exists a 2-step arrow $r : A \downarrow^2 x$ such that, for every 2-step arrow $f : A \downarrow^2 x$, there exists a zigzag

$$f = f_0 \xrightarrow{s_0}{}^{s} g_0 \,{}^{t}\!\xleftarrow{t_0} f_1 \xrightarrow{s_1}{}^{s} \cdots \xrightarrow{s_{m-1}}{}^{s} g_{m-1} \,{}^{t}\!\xleftarrow{t_{m-1}} f_m = r,$$

where g_i's are source arrows into x, s_i's are source arrows in $A \downarrow x$, and t_i's are target arrows in $A \downarrow x$.

Tree structures on sources





The pasting diagram on the left has the tree structure on the right. Dots and lines in the tree correspond to 2-dimensional cells and 1-dimensional cells, respectively, in the pasting diagram.

Opetopic set axioms

A couple of global axioms.

Axiom (O7)

For every target arrow $f : y \to^t x$ in A and object $z : A$ of degree $\leq \mathbf{deg}(y) - 2$, the postcomposition map $f_! : \mathbf{Arr}_A(z, y) \to \mathbf{Arr}_A(z, x)$ is injective.

Axiom (O8)

For every $k \geq 3$, every k-step arrow $y \to^k x$ in A factors as $f \circ g$ such that f is a $(k-1)$-step arrow and g is a 1-step arrow.

Opetopes

Definition

An **opetope** is an opetopic set in which a terminal object exists.

Let $\mathbf{OSet}$ denote the category whose

- objects are small opetopic sets;
- morphisms are functors preserving degrees, source arrows, and target arrows.

Let $\mathbb{O} \subset \mathbf{OSet}$ denote the full subcategory spanned by opetopes.

Results

- $\mathbf{OSet} \simeq \mathbf{Psh}(\mathbb{O})$.
- Equivalence with the polynomial monad definition given by Kock, Joyal, Batanin, and Mascari (2010).
- Presentation of the category of opetopes equivalent to one given by Ho Thanh (2021).

Outline

Introduction

Opetopic sets and opetopes

Opetopic sets as presheaves

Comparison with the polynomial monad definition

Normalization

Proposition

In any opetopic set, every $f : x \to^k y$ for $k \geq 2$ uniquely factors into k 1-step arrows

$$g_1 \circ \ldots \circ g_k$$

such that

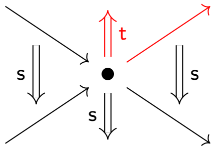
- $g_1, \ldots, g_{k-2}$ *are target arrows;*
- (g_{k-1}, g_k) *is homogeneous.*

Proof.

Factor f into k 1-step arrows in any way. Rewrite according to Axioms O4 and O5. It terminates! □

Normalization

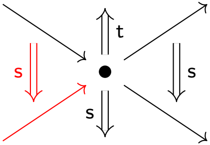
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, "walking around counterclockwise".

Normalization

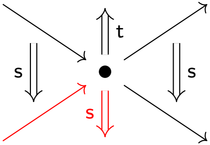
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, “walking around counterclockwise”.

Normalization

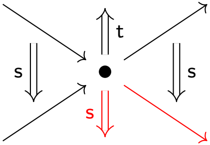
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell $(\bullet)$ into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, “walking around counterclockwise”.

Normalization

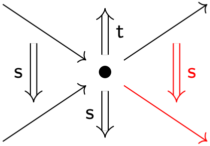
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, "walking around counterclockwise".

Normalization

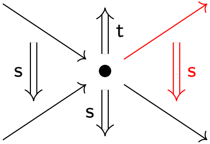
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, "walking around counterclockwise".

Normalization

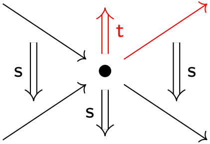
Consider some part of a 3-opetope.



- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, "walking around counterclockwise".

Normalization

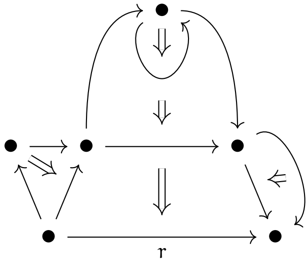
Consider some part of a 3-opetope.

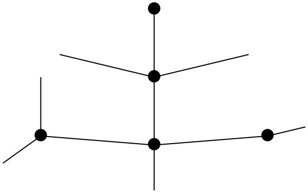


- Exactly one way to embed the 0-cell ($\bullet$) into the 3-cell as a source of a source of the target, which is the normal form.
- A canonical path to the normal form from any other position, "walking around counterclockwise".

Tree structures on sources

Let $A \downarrow^{s} x \subset A \downarrow^{1} x$ denote the set of source arrows into x. Then $A \downarrow^{s} x$ is the set of nodes of a tree.





Local rigidity

Proposition

Let $F_1, F_2 : A \to A'$ *be morphisms of opetopic sets,* $x : A$*, and* $x' : A'$ *such that* $F_1(x) = F_2(x) = x'$*. Then*

$$F_1 \downarrow x, F_2 \downarrow x : A \downarrow x \to A' \downarrow x'$$

are identical.

Proof.

By normalization, it suffices to see $F_1 \downarrow^s x = F_2 \downarrow^s x$. This holds because there is at most one map preserving the tree structure on sources. □

Local rigidity

Proposition

Let $F : A \to A'$ *be a morphism of opetopic sets and* $x : A$. *Then*

$$F \downarrow x : A \downarrow x \to A' \downarrow F(x)$$

is an equivalence.

Proof.

By normalization, it suffices to see $F \downarrow^s x : A \downarrow^s x \simeq A' \downarrow^s F(x)$. Use the tree structure on sources. □

Local rigidity

Corollary

$\mathbb{O}$ *is a gaunt category.*

Corollary

Every morphism of opetopic sets is a discrete fibration.

Corollary

$\mathbf{OSet} \downarrow A \simeq \mathbf{Psh}(A)$ *for every* $A : \mathbf{OSet}$.

Local finiteness

Proposition

Let A be an opetopic set. Then $A \downarrow x$ is finite for every $x : A$.

Proof.

By normalization and Axiom O1. □

Corollary

Every opetope is finite.

Corollary

$\mathbb{O}$ *is small.*

The opetopic set of opetopes

We extend $\mathbb{O}$ to a preoeptopic set.

- $\mathbf{deg}_{\mathbb{O}}(A) \equiv \mathbf{deg}_{A}(*_A)$, where $*_A : A$ is the terminal object.
- $F : A' \to A$ is a source/target arrow if $F(*_{A'}) \to *_A$ is a source/target arrow.

The opetopic set of opetopes

Proposition

Let A be an opetopic set. The morphism of preopetopic sets

$$\begin{aligned} A &\to \mathbb{O} \downarrow A \\ x &\mapsto A \downarrow x \end{aligned}$$

is an equivalence.

Proof.

The inverse sends $F : B \to A$ to $F(*_B)$. □

Corollary

$\mathbb{O}$ is an opetopic set (because every slice $\mathbb{O} \downarrow A \simeq A$ satisfies the axioms).

The terminal opetopic set

Proposition

$\mathbb{O} : \mathbf{OSet}$ *is the terminal object.*

Proof.

$(x \mapsto A \downarrow x) : A \to \mathbb{O}$ is the unique morphism. □

Corollary

$\mathbf{OSet} \simeq \mathbf{Psh}(\mathbb{O})$ *(special case of* $\mathbf{OSet} \downarrow A \simeq \mathbf{Psh}(A)$*).*

Outline

Polynomials

A **polynomial** P on I consists of maps of sets

$$I \xleftarrow{s_P} E(P) \xrightarrow{p_P} B(P) \xrightarrow{t_P} I.$$

- $B(P)$ is a set of “typed operators”.
- $t_P(b)$ is the output type.
- The fiber $E(P)_b$ is the set of inputs.
- $s_P(e)$ is the input type.

A **polynomial monad** is a polynomial in which “operators can be composed”.

The polynomial monad definition of opetopes

By Kock, Joyal, Batanin, and Mascari (2010).

- For every polynomial monad P on I, there is a polynomial monad P^{+} on $\mathbf{B}(P)$, called the **Baez-Dolan construction**.
- The set of KJBM n-opetopes $\mathbb{O}_n^{\mathrm{KJBM}}$ and the polynomial monad $\mathbf{Z}_n$ on $\mathbb{O}_n^{\mathrm{KJBM}}$ are defined by
  - $\mathbb{O}_0^{\mathrm{KJBM}} \equiv \mathbf{1}$;
  - $\mathbf{Z}_0 \equiv (\mathbf{1} = \mathbf{1} = \mathbf{1} = \mathbf{1})$;
  - $\mathbb{O}_{n+1}^{\mathrm{KJBM}} \equiv \mathbf{B}(\mathbf{Z}_n)$;
  - $\mathbf{Z}_{n+1} \equiv \mathbf{Z}_n^{+}$.

Equivalence with the polynomial monad definition

Theorem

$\mathbb{O}_n \simeq \mathbb{O}_n^{\mathrm{KJBM}}$

Proof sketch.

Let $\mathbf{Y}_n$ be the polynomial on $\mathbb{O}_n$

$$\mathbb{O}_n \xleftarrow{s_{\mathbf{Y}_n}} \mathsf{E}(\mathbf{Y}_n) \to \mathbb{O}_{n+1} \xrightarrow{t} \mathbb{O}_n,$$

where the fiber $\mathsf{E}(\mathbf{Y}_n)_A$ is $\mathbb{O}_{n+1} \downarrow^{s} A$. Show $\mathbf{Y}_0 \simeq \mathbf{Z}_0$ and $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$. □

There are two compositional structures on pasting diagrams, **substitution** and **grafting**. The polynomial monad structure on $\mathbf{Y}_n$ is defined by substitution, and the equivalence $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$ is proved by interaction between substitution and grafting.

Categorical equivalence

Ho Thanh (2021) gives a definition of the category of opetopes, whose objects are the KJBM opetopes, by generators and relations. Our $\mathbb{O}$ has the following presentation, which is equivalent to Ho Thanh's.

Proposition

The category $\mathbb{O}$ is presented by:

- **Generators** *all the 1-step arrows in $\mathbb{O}$;*
- **Relations** *all the equations $f_1 \circ g_1 = f_2 \circ g_2$ that hold in $\mathbb{O}$ such that (f_1, g_1) is heterogeneous and (f_2, g_2) is homogeneous.*

This also holds for the underlying category of any opetopic set A.

Proof.

By normalization. □

Summary

- Opetopes and opetopic sets are encoded as categories of cells.
- No need to care about loops.
- Equivalent to the polynomial monad definition (and other definitions).

References I

- John C. Baez and James Dolan (1998). "Higher-dimensional algebra. III. n-categories and the algebra of opetopes". In: *Adv. Math.* 135.2, pp. 145–206. DOI: 10.1006/aima.1997.1695.
- Clark Barwick and Christopher Schommer-Pries (2021). "On the unicity of the theory of higher categories". In: *J. Amer. Math. Soc.* 34.4, pp. 1011–1058. DOI: 10.1090/jams/972.
- Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram (2022). "Type theoretical approaches to opetopes". In: *Higher Structures* 6.1, pp. 80–181.
- Amar Hadzihasanovic (2020). "A Combinatorial-Topological Shape Category for Polygraphs". In: *Applied Categorical Structures* 28, pp. 419–476. DOI: 10.1007/s10485-019-09586-6.

References II

- Claudio Hermida, Michael Makkai, and John Power (2002). "On weak higher-dimensional categories. I. 3". In: *J. Pure Appl. Algebra* 166.1-2, pp. 83–104. DOI: 10.1016/S0022-4049(01)00014-7.
- Cédric Ho Thanh (2021). *The equivalence between many-to-one polygraphs and opetopic sets.* arXiv: 1806.08645v4.
- Martin Hofmann and Thomas Streicher (1998). "The groupoid interpretation of type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)*. Vol. 36. Oxford Logic Guides. New York: Oxford Univ. Press, pp. 83–111. DOI: 10.1093/oso/9780198501275.003.0008.
- Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari (2010). "Polynomial functors and opetopes". In: *Adv. Math.* 224.6, pp. 2690–2737. DOI: 10.1016/j.aim.2010.02.012.

References III

- Louise Leclerc (2024). *Two equivalent descriptions of opetopes: in terms of zoom complexes and of partial orders*. arXiv: 2409.02744v1.
- Tom Leinster (2004). *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press. DOI: 10.1017/CBO9780511525896. arXiv: math/0305049v1.
- Peter McMullen and Egon Schulte (2002). *Abstract Regular Polytopes*. Encyclopedia of Mathematics and its Applications. Cambridge University Press. DOI: 10.1017/CBO9780511546686.
- The Univalent Foundations Program (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study. URL: http://homotopytypetheory.org/book/.