Differential Bundles in the Opposite Category of Schemes

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TL;DR: The Punchline

The Main Goal

We will explain how, for any scheme S and any S-scheme X there is an equivalence

$$\mathbf{DBun}_{\mathbf{Sch}^{\mathrm{op}}_{/S}}(X) \simeq \mathbf{QCoh}(X)$$

for $\mathbf{DBun}(X)$ the category of differential bundles over X (in $\mathbf{Sch}^{\mathrm{op}}_{/S}$) and for $\mathbf{QCoh}(X)$ the category of quasi-coherent sheaves on X.

A Second Goal

We also want to introduce the motto/theme: to work with differential bundles in schemes is ultimately to work with affine morphisms and with the relative spectrum functor so that we can pretend the world is one of affine schemes.

Outline

The Plan

1. Our Friend the Tangent Category
2. Our Other Friend the Differntial Bundle
3. Scheme-theoretic Background
4. Quasi-Coherent Sheaves as Differential Bundles
5. Differential Bundles as Quasi-Coherent Sheaves
6. The Equivalence!

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Why Tangent Categories

The Idea Behind Tangent Categories

Tangent categories provide a minimal categorical setting in which to do differential-geometric reasoning. They were first introduced by Rosický in [Rosický1984] and later refined and reintroduced by Cockett and Cruttwell in [CockettCruttwell2014].

Tangent Categories

Definition: Tangent Category [CockettCruttwell2014]

A tangent category $(\mathscr{C}, \mathbb{T})$ consists of a category $\mathscr{C}$ a tangent structure $\mathbb{T}$. This tangent structure $\mathbb{T}$ is a 6-tuple $(T, p, 0, \text{add}, \ell, c)$ where:

- An endofunctor $T : \mathscr{C} \to \mathscr{C}$ called the tangent functor.
- A natural transformation $p : T \Rightarrow \text{id}_{\mathscr{C}}$ called the bundle projection.
- A natural transformation $0 : \text{id}_{\mathscr{C}} \Rightarrow T$ called the zero transformation.
- A natural transformation $\text{add} : T_2 \Rightarrow T$ called the addition for $T_2X := TX \times_X TX$.
- A natural transformation $\ell : T \Rightarrow T^2$ called the vertical lift.
- A natural transformation $c : T^2 \Rightarrow T^2$ called the canonical flip.
- Subject to coherences (omitted here)!

Remarks

- We will often omit the mention of the tangent structure $\mathbb{T}$ when saying that something is a tangent category.
- If $(\mathscr{C}, \mathbb{T})$ is a tangent category such that $\mathscr{C}$ has finite products and T preserves these finite products, then we say $\mathscr{C}$ is a Cartesian tangent category.
- The pullback T_2X is the categorical analogue of the space of two tangent vectors anchored at the same base point.
- By [LemayVooys2026], every tangent functor T is both limit and colimit reflecting.

Examples 1

The Trivial Example

Every category $\mathscr{C}$ is a tangent category with trivial tangent structure $\mathbb{I} = (\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$. It is a Cartesian tangent category if and only if $\mathscr{C}$ has finite products.

Category with Finite Biproducts

If $\mathscr{C}$ is any category with finite biproducts then $\mathscr{C}$ is a Cartesian tangent category with tangent functor given by, for $X : \mathscr{C}_0$ and $f : \mathscr{C}_1$, $TX := X \oplus X$ and $Tf := f \oplus f$. This includes **Rel**, every additive category, and **CMon**.

Examples 2

Smooth Manifolds

Let **SMan** denote the category of smooth manifolds. Then **SMan** is a Cartesian tangent category with the usual tangent functor $TX := \sqcup_{x:X} T_x X$.

Slice Categories

Let $(\mathscr{C}, \mathbb{T})$ be a tangent category. Then $\mathscr{C}_{/X}$ is a tangent category with tangent functor given by

$$T(A \xrightarrow{f} X) := TA \xrightarrow{Tf} TX \xrightarrow{p_X} X$$

This example is a Cartesian tangent category when $\mathscr{C}$ admits finite pullbacks over X and when $T(A \times_X B) \cong TA \times_X TB$.

Examples 3

Cartesian Differential Categories

Let $\mathscr{C}$ be a Cartesian Differential Category (CDC) with differential combinator:

$$\frac{f : X \to Y}{D[f] : X \times X \to Y}$$

Then $\mathscr{C}$ is a Cartesian tangent category with tangent functor given by $TX := X \times X$ and $Tf := \langle f \circ \pi_0, D[f] \rangle$.

The Lawvere Theore of Euclidean Spaces

A special case of the above is the CDC **Euc** of Euclidean spaces. This is the finitary Lawvere theory with objects $n : \mathbb{N}$ and with morphisms determined by $\mathbf{Euc}(n, m) := \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Examples 4

Commutative Algebras

Let R be a commutative rig[a] and write $\mathbf{CAlg}_R = R \downarrow \mathbf{CRig}$ for the category of commutative R-algebras. Then $\mathbf{CAlg}_R$ is a Cartesian tangent category with tangent functor given by sending A to its associated rig of dual numbers $TA := A[\varepsilon] \cong A[x]/(x^2)$.

[a] A rig, also known as a semiring, is a ring without negatives.

Commutative Algebras, Opposite

Let R be a commutative rig. Then $\mathbf{CAlg}_R^{\mathrm{op}}$ is a Cartesian tangent category with tangent functor

$$TA := \mathrm{Sym}_A \, \Omega_{A/R}$$

where $\Omega_{A/R}$ is the A-module of relative Kähler differentials.

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Why Differential Bundles

Idea

In many ways, differential bundles (defined by Cockett and Cruttwell in [CockettCruttwell2018]) in a tangent category are simultaneously the analogue of vector bundles in differential geometry and of modules in algebraic geometry. They are given effectively as special internal commutative monoids (q, ζ, σ) in $\mathscr{C}_{/X}$ for which the monoidal operation "interacts well" with the monoidal operation on the tangent bundle of (q, ζ, σ).

Differential Bundles: Definition Part 1

Definition

Let $\mathscr{C}$ be a tangent category. A differential bundle (over X) in $\mathscr{C}$ is a quadruple

$$\mathsf{q} = \begin{pmatrix} E & X & E_2 & E \\ \downarrow q, & \downarrow \zeta, & \downarrow \sigma, & \downarrow \lambda \\ X & E & E & TE \end{pmatrix}$$

for which the following hold:

- For all $n : \mathbb{N}$, the n-th wide pullback E_n of $q : E \to X$ exists and is preserved by T^m for all $m : \mathbb{N}$.
- The triple (q, ζ, σ) is a commutative monoid internal to $\mathscr{C}_{/X}$.
- Various coherences, universal properties, and conditions we omit here.

Morphisms of Differential Bundles

Definition

A morphism of differential bundles q $\to$ r is a pair of maps we write as

$$\begin{pmatrix} \varphi \\ f \end{pmatrix} : \begin{pmatrix} E & & E \\ \downarrow q & , \cdots , & \downarrow \lambda \\ X & & TE \end{pmatrix} \to \begin{pmatrix} F & & F \\ \downarrow r & , \cdots , & \downarrow \mu \\ Y & & TF \end{pmatrix}$$

for $f : X \to Y$ and $\varphi : E \to F$ which make the diagram below commute:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Definition

A map of differential bundles

$$\begin{pmatrix} \varphi \\ f \end{pmatrix} : \begin{pmatrix} E & & E \\ \downarrow q & \cdots & \downarrow \lambda \\ X & & TE \end{pmatrix} \to \begin{pmatrix} F & & F \\ \downarrow r & \cdots & \downarrow \mu \\ Y & & TF \end{pmatrix}$$

is linear when the diagram below also commutes:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mu} \\ TE & \xrightarrow[T\varphi]{} & TF \end{array}$$

Categories of Differential Bundles

Definitions

Let $(\mathscr{C}, \mathbb{T})$ be a tangent category. We write $\mathbf{DBun}(X)$ for the category of differntial bundles over a fixed $X : \mathscr{C}_0$ and linear maps of the form $\begin{pmatrix} \varphi \\ \mathrm{id}_X \end{pmatrix}$.

Remark

If context may be needed, we will write $\mathbf{DBun}_{\mathscr{C}}(X) = \mathbf{DBun}_{(\mathscr{C},\mathbb{T})}(X)$ to emphasise in which tangent category with which we consider differential bundles.

Remark on Linearity

Remark on Linearity

By [CockettCruttwell2018], if $\binom{\varphi}{f}$ is a linear morphism of differential bundles, then $\binom{\varphi}{f}$ is linear the usual sense. That is, f and φ commute appropriately with the sum and unit operations.

Examples of Differential Bundles 1

Differential Bundles in **Euc**

Let $n : \mathbb{N}$. Then in the CDC **Euc** of Euclidean spaces and smooth functions we have that $\mathbf{DBun}(n) \simeq \mathbb{R} - \mathbf{Vect}_{\text{f.d.}}$.

Differential Bundles in Smooth Paracompact Hausdorff Manifolds

Let $\mathbf{SMan}_{\text{Pcomp,Hdorff}}$ be the tangent category of paracompact Hausdorff smooth manifolds. Then by [MacAdam2021] we have that for all paracompact Hausdorff smooth manifolds X, $\mathbf{DBun}(X) \cong \mathbf{Vec}(X)$.

Examples of Differential Bundles 2

Commutative Algebras

Let R be a commutative rig. By [CruttwellLemay2023], for any R-algebra A there is an equivalence of categories

$$\mathbf{DBun}_{\mathbf{CAlg}_R}(A) \simeq A\text{-}\mathbf{Mod}.$$

Commutative Algebras, Opposite

Let R be a commutative rig. By [CruttwellLemay2023], for any R-algebra A there is an equivalence of categories

$$\mathbf{DBun}_{\mathbf{CAlg}_R^{\mathrm{op}}}(A) \simeq A\text{-}\mathbf{Mod}^{\mathrm{op}}.$$

The Most Important Example for Today

S-Schemes

Let S be a scheme and let X be an S-scheme. Then in the tangent category $\mathbf{Sch}_{/S}$ we have from [CruttwellLemay2023] that there is an equivalence

$$\mathbf{DBun}_{\mathbf{Sch}_{/S}}(X) \simeq \mathbf{QCoh}(X)^{\mathrm{op}}.$$

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The Tangent Structures for Rings

A Tangent Categorical Result

We saw in our examples that if R is a commutative ring then there are tangent structures on $\mathbf{CAlg}_R = R \downarrow \mathbf{Cring}$ and $\mathbf{CAlg}_R^{\mathrm{op}}$. The tangent functor for $\mathbf{CAlg}_R$ is the "ring of dual numbers" functor

$$(-)[\varepsilon] : \mathbf{CAlg}_R \to \mathbf{CAlg}_R, \ A \mapsto A[\varepsilon]$$

while the tangent functor for $\mathbf{CAlg}_R^{\mathrm{op}}$ is the tangent space functor

$$T_{(-)/R} : \mathbf{CAlg}_R^{\mathrm{op}} \to \mathbf{CAlg}_R^{\mathrm{op}}, \ A \mapsto \mathrm{Sym}_A \Omega_{A/R}.$$

The Relationship

The important relationship between the tangent structures on $\mathbf{CAlg}_R$ and $\mathbf{CAlg}_R^{\mathrm{op}}$ is that they are adjoint and hence dual in a precise sense. That is, they arise from the adjunction

$$\mathbf{CAlg}_R \underset{(-)[\varepsilon]}{\overset{\mathrm{Sym}_{(-)}\,\Omega_{(-)/R}}{\rightleftarrows}} \mathbf{CAlg}_R \qquad \text{with } \dashv$$

the fact that the right adjoint is a tangent functor, and from the dual tangent structure construction of [CockettCruttwell2014, Proposition 5.16].

The Goal

A Change in Perspective

We now want to show how, upon schemifying the story, we have an adjunction

$$\mathbf{Sch}_{/S} \underset{T_{(-)/S}}{\overset{(-)[\varepsilon]}{\rightleftarrows}} \mathbf{Sch}_{/S} \qquad (\perp)$$

and how to see the ways the tangent functor $(-)[\varepsilon]$ behaves on $\mathbf{Sch}^{\mathrm{op}}_{/S}$.

Moving from Rings to Schemes

Core Idea

When describing the tangent structure on $\mathbf{Sch}_{/S}$ which mimics the story for rings, we need to extend the same basic ideas as the affine case by way of the relative spectrum functor and affine morphisms. For the nitty-gritty details of this, see [Vooys2026].

Quasi-Coherent Sheaves on Affine Schemes

Let A be a commutative ring and let M be an A-module. Define the quasi-coherent sheafification of M to be the sheaf $\widetilde{M}$ determined by, for all $f : A$

$$\widetilde{M}[D(f)] := M \otimes_A A[f^{-1}] = M[f^{-1}]$$

where $D(f) = \{\mathfrak{p} : |\operatorname{Spec} A| \mid f \notin \mathfrak{p}\}$ is the basic open generated by f.

Definition

Let $\operatorname{Spec} A$ be an affine scheme. A sheaf $\mathscr{F}$ of $\mathcal{O}_A$ is a quasi-coherent sheaf on X if there is an A-module M such that $\mathscr{F} \cong \widetilde{M}$.

Quasi-Coherent Sheaves

Let $X = (|X|, \mathcal{O}_X)$ be a scheme. A quasi-coherent sheaf on X is a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules such that:

- There is an affine open cover $\{\gamma_i : \operatorname{Spec} A_i \to X \mid i : I\}$, for each $i : I$ there is an A_i-module M_i for which

$$\gamma_i^* \mathscr{F} \cong \widetilde{M_i}.$$

- For all $i, j : I$, the isomorphisms $\gamma_i^* \mathscr{F} \cong \widetilde{M_i}$ are compatible (up to isomorphism) with restrictions to the intersections $\operatorname{Spec} A_i \times_X \operatorname{Spec} A_j$.

We write $\mathbf{QCoh}(X)$ for the category of quasi-coherent sheaves on X.

Quasi-Coherent Sheaves of Algebras

When $\mathcal{O}_X$-algebras are Quasi-Coherent

Let $X = (|X|, \mathcal{O}_X)$ be a scheme. A sheaf $\mathscr{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras when:

- $\mathscr{A}$ is a commutative algebra over $\mathcal{O}_X$ in $\mathbf{Shv}(X)$.
- $\mathscr{A}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules.

We write $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$ for the category of quasi-coherent sheaves of $\mathcal{O}_X$-algebras.

Relative Symmetric Algebras

There is a forgetful functor $\underline{\text{Forget}} : \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X}) \to \mathbf{QCoh}(X)$. It has a left adjoint we call the relative symmetric algebra functor $\underline{\text{Sym}}_{\mathcal{O}_X}(-)$.

Why Quasi-Coherent Sheaves Get Used

Grothendieck's Characterisation of $\mathbf{QCoh}(X)$ for Affine Schemes

For any commutative ring A, $\mathbf{QCoh}(\operatorname{Spec} A) \simeq A\text{-}\mathbf{Mod}$.

Theorem (Folklore; cf. [Vooys2026])

Let X be a scheme and let $\mathbf{AOpen}(X)$ be the poset of affine open subschemes of X and let $(-)\text{-}\mathbf{Mod} : \mathbf{AOpen}(X)^{\mathrm{op}} \to \mathfrak{Cat}$ be the pseudofunctor which sends an affine open subscheme $\operatorname{Spec} A$ to $A\text{-}\mathbf{Mod}$ and sends an inclusion $j : \operatorname{Spec} B \to \operatorname{Spec} A$ to the extension of scalars functor $(-) \otimes_A B : A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}$. Then there is a pseudolimit decomposition

$$\mathbf{QCoh}(X) \simeq \operatorname*{pslim}_{U : \mathbf{AOpen}(X)_0} \mathcal{O}_X(U)\text{-}\mathbf{Mod}.$$

Why Quasi-Coherent Sheaves of Algebras Get Used

Sanity Check of $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$ for Affine Schemes

For any commutative ring A, $\mathbf{QCoh}(\operatorname{Spec} A, \mathbf{CAlg}_{\mathcal{O}_A}) \simeq \mathbf{CAlg}_A$.

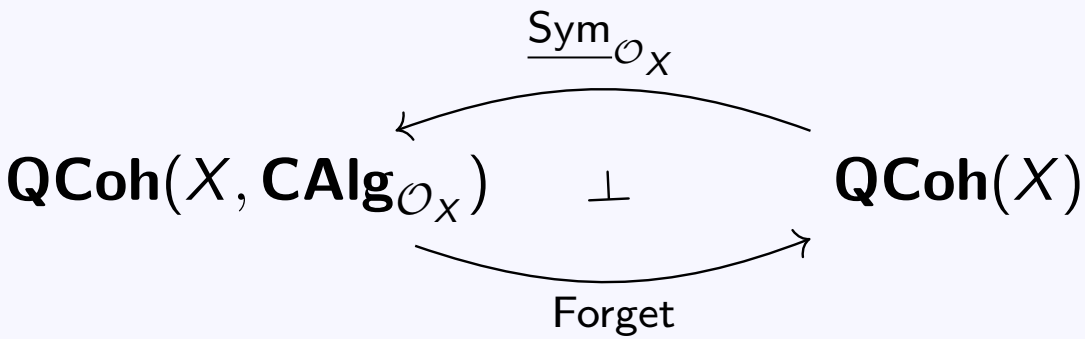
Theorem (Less Well-Known Folklore)

Let X be a scheme and let $\mathbf{AOpen}(X)$ be the poset of affine open subschemes of X and let $\mathbf{CAlg}_{(-)} : \mathbf{AOpen}(X)^{\mathrm{op}} \to \mathfrak{Cat}$ be the pseudofunctor which sends an affine open subscheme $\operatorname{Spec} A$ to $\mathbf{CAlg}_A$ and sends an inclusion $j : \operatorname{Spec} B \to \operatorname{Spec} A$ to the functor $(-) \otimes_A B : \mathbf{CAlg}_A \to \mathbf{CAlg}_B$. Then there is a pseudolimit decomposition

$$\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X}) \simeq \operatorname*{pslim}_{U \in \mathbf{AOpen}(X)_0} \mathbf{CAlg}_{\mathcal{O}_X(U)}.$$

A Neat Consequence

The adjunction



is the pseudolimit extension of the local adjunctions

$$\mathrm{Sym}_{\mathcal{O}_X(U)} \dashv \mathrm{Forget}_{\mathcal{O}_X(U)}\,.$$

This allows us to work with quasi-coherent sheaves and quasi-coherent sheaves of algebras much as we would work with rings and modules. From here, however, we move to discuss affine morphisms.

Affine Structure Morphisms

Definition

A morphism of schemes $f : X \to Y$ is affine if for any affine open subscheme $\gamma : V \to Y$, the pullback $X \times_Y V$ is an affine scheme as well.

Fact

The class of affine morphisms is a pre-compositional ideal in $\mathbf{Sch}_{/S}$. That is, given any composite $g \circ f$ if g is affine and $g \circ f$ is affine, then f is affine as well.

Definition

We write $\mathbf{Aff}_S$ for the category of schemes $f : X \to S$ for which the structure map f is affine.

The Relative Spectrum Functor

Theorem [EGA2]

For any scheme X, there is an equivalence of categories

$$\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})^{\mathrm{op}} \simeq \mathbf{Aff}_X$$

with the functor $\mathbf{Aff}_X \to \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})^{\mathrm{op}}$ given on objects by $f : Y \to X \mapsto f_* \mathcal{O}_Y$.

Definition

Let X be a scheme. The relative spectrum functor $\underline{\mathrm{Spec}}_X$ is the equivalence:

$$\underline{\mathrm{Spec}}_X : \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{Aff}_X$$

An Important Consequence

If $f : X \to S$ is an affine morphism of schemes, then the pushforward functor $f_* : \mathbf{QCoh}(X) \to \mathcal{O}_S\text{-}\mathbf{Mod}$ factors through $\mathbf{QCoh}(S)$. That is, f_* types as

$$f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(S).$$

A Further Consequence

For any morphism of schemes $f : X \to S$, f is affine if and only if $f_* \mathcal{O}_X$ is quasi-coherent over S.

Towards Differential Bundles for Schemes

Why do we Care?

The functor $\underline{\mathrm{Spec}}_X$ is the tool that lets us work with schemes as if everything in sight was affine! Much of the technical work behind why any of the tangent category theory of schemes happens in the first place is because of affine morphisms. In fact:

Using Affine Structure Morphisms

In [CruttwellLemay2023], Cruttwell and Lemay showed that for any scheme S and any S-scheme X, there is an equivalence of categories

$$\mathbf{QCoh}(X)^{\mathrm{op}} \simeq \mathbf{DBun}_{\mathbf{Sch}_{/S}}(X)$$

by making heavy use of affine morphisms.

The Tangent Structure for Schemes

Definition

Let S be a scheme and let X be an S-scheme. The sheaf of relative Kähler differentials is the sheaf $\Omega_{X/S}$ which co-represents the functor

$$\mathrm{Der}_S(\mathcal{O}_X, -) : \mathbf{QCoh}(X) \to \mathcal{O}_X(X)\text{-}\mathbf{Mod}, \quad \mathscr{F} \mapsto \mathrm{Der}_S(\mathcal{O}_X, \mathscr{F}).$$

The Tangent Functor for Schemes; cf. [CruttwellLemay2023]

Let S be a scheme and let X be an S-scheme. Then the tangent functor $T_{(-)/S} : \mathbf{Sch}_{/S} \to \mathbf{Sch}_{/S}$ is defined by

$$T_{X/S} := \underline{\mathrm{Spec}}_X \left(\underline{\mathrm{Sym}}_{\mathcal{O}_X} \Omega_{X/S} \right).$$

An Left Adjoint to $\mathcal{T}_{(-)/S}$

Observation

We can use the adjoints which define the tangent structures on rings, do some Zariski descent/psuedolimit things, and be careful with our directionality of morphisms to derive that there is an adjunction:

$$\mathbf{Sch}_{/S} \underset{\mathcal{T}_{(-)/S}}{\overset{(-)[\varepsilon]}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Sch}_{/S}$$

As before, this means that the functor $(-)[\varepsilon] : \mathbf{Sch}^{\mathrm{op}}_{/S} \to \mathbf{Sch}^{\mathrm{op}}_{/S}$ is a tangent functor! But what is this functor?

The Scheme of Dual Numbers Functor

Sheaf of Dual Numbers Functor

Let X be a scheme. Define the functor

$$(-)[\varepsilon] : \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X}) \to \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$$

by letting $\widetilde{\mathcal{O}_X[\varepsilon]}$ be the sheaf where, for any affine open $\gamma : \operatorname{Spec} A \to X$, $\gamma^* \mathcal{O}_X[\varepsilon] := \widetilde{A[\varepsilon]}$.

Scheme of Dual Numbers Functor

Let S be a scheme. We define the scheme of dual numbers functor $(-)[\varepsilon] : \mathbf{Sch}_{/S} \to \mathbf{Sch}_{/S}$ (and its formal dual) by the assignment on objects

$$X \mapsto \underline{\operatorname{Spec}}_X(\mathcal{O}_X[\varepsilon]).$$

This is the tangent functor on $\mathbf{Sch}^{\mathrm{op}}_{/S}$.

Where We're Going

Keeping the Goal in Sight

Our goal, now that we know our tangent category, is to show how to take quasi-coherent sheaves and turn them into differential bundles in $\mathbf{Sch}^{\mathrm{op}}_{/S}$ and vice-versa.

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Motivating our Tool at Hand

The Motivation

When Cruttwell and Lemay showed that $\mathbf{DBun}_{\mathbf{CAlg}_R}(A) \simeq A\text{-}\mathbf{Mod}$ in [CruttwellLemay2023], they used a trick which allowed them to take A-modules and view them as A-algebras via the use of nil-square extensions. We review this now to use it in the scheme-y case.

Split Nil-Square Extensions of Rings

Definition

Let A be a commutative ring and let M be an A-module. Define the nil-square extension of A by M to be the A-algebra $A \ltimes M$ given as follows:

- Underlying set: The underlying A-module of $A \ltimes M$ is $A \oplus M$.
- Multiplication: For all $a, b : A$ and for all $x, y : M$ we define the multiplication via the ring of dual numbers multiplication:

$$(a, x)(b, y) := (ab, ay + bx).$$

Definition

We write $A \ltimes (-) : A\text{-}\mathbf{Mod} \to \mathbf{CAlg}_A$ for the functor which sends an A-module to the nil-square extension $A \ltimes M$.

Some Neat Facts About Nil-Square Extensions

Here are some facts about the functor $A \ltimes (-) : A\text{-}\mathbf{Mod} \to \mathbf{CAlg}_A$.

- The algebra $A \ltimes M$ is an augmented A-algebra. The augmentation map $A \ltimes M \to A$ is the first projection.
- The functor $A \ltimes (-)$ factors through the category of augmented A-algebras.
- For any module M, the natural map $|\mathrm{Spec}(A \ltimes M)| \to |\mathrm{Spec}\, A|$ is a homeomorphism.
- If B is an augmented A-algebra with nil-square kernel, then $B \cong A \ltimes M$ for some A-module M.
- The equivalence $A\text{-}\mathbf{Mod} \to \mathbf{DBun}_{\mathbf{CAlg}_R}(A)$ sends the module M to a differential bundle whose projection is $\pi_0 : A \ltimes M \to A$.
- The functor $A \ltimes (-)$ preserves binary pullbacks and initial objects. It preserves the terminal object if and only if A is the zero ring.

Nil-Square Extensions of Quasi-Coherent Sheaves

Definition

Let X be a scheme. If $\mathscr{F}$ is a quasi-coherent sheaf on X, we define the sheaf $\mathcal{O}_X \ltimes \mathscr{F}$ by setting, for all opens U of X,

$$(\mathcal{O}_X \ltimes \mathscr{F})(U) := \mathcal{O}_X(U) \ltimes \mathscr{F}(U).$$

This gives a functor $\mathcal{O}_X \ltimes (-) : \mathbf{QCoh}(X) \to \mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$.

Definition

Let X be a scheme. We define the functor $X \ltimes (-) : \mathbf{QCoh}(X)^{\mathrm{op}} \to \mathbf{Aff}_X$ by

$$\mathscr{F} \mapsto \underline{\mathrm{Spec}}_X(\mathcal{O}_X \ltimes \mathscr{F}).$$

Facts About the Nil-Square Extension Functor

Four Facts

- There is a functor $\mathcal{O}_X \ltimes (-) : \mathbf{QCoh}(X) \to \mathbf{QCoh}(X, \mathcal{O}_X)$ given by $\mathscr{F} \mapsto \mathcal{O}_X \ltimes \mathscr{F}$.
- The functor $\mathcal{O}_X \ltimes (-)$ preserves binary pullbacks and initial objects. It preserves the terminal object if and only if $X \cong \operatorname{Spec} 0$.
- The functor $\mathcal{O}_X \ltimes (-)$ is fully faithful.
- For any quasi-coherent sheaf $\mathscr{F}$, the map $|X \ltimes \mathscr{F}| \to |X|$ is a homeomorphism.

An Important Technical Fact

A Weird Technical Result

Let S be a scheme and let X be an S-scheme. For any pullback $\mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ in $\mathbf{QCoh}(X)$, the relative spectrum

$$\underline{\mathrm{Spec}}_X(\mathcal{O}_X \ltimes (\mathscr{F} \times_{\mathscr{H}} \mathscr{G})) = X \ltimes (\mathscr{F} \times_{\mathscr{H}} \mathscr{G})$$

is the pushout of

$$X \ltimes \mathscr{F} \leftarrow X \ltimes \mathscr{H} \rightarrow X \ltimes \mathscr{G}$$

in $\mathbf{Sch}_{/S}$. (This fact is strangely difficult to prove)

The Functor $\mathbf{QCoh}(X) \to \mathbf{DBun}(X)$ Part 1

Defining The Differential Bundle

We want to define the functor $\mathbb{D} : \mathbf{QCoh}(X) \to \mathbf{DBun}(X)$ by showing that $\mathscr{F} \mapsto X \ltimes \mathscr{F}$ gives us all the structure we need to define $(q_{\mathscr{F}}, \zeta_{\mathscr{F}}, \sigma_{\mathscr{F}}, \lambda_{\mathscr{F}})$. The calculus of formal opposites gives for any $\mathscr{F} : \mathbf{QCoh}(X)_0$,

$$\frac{\dfrac{\pi_0 : \mathcal{O}_X \ltimes \mathscr{F} \to \mathcal{O}_X \text{ in } \mathbf{QCoh}(X)}{\underline{\mathrm{Spec}}_X(\pi_0) : X \to X \ltimes \mathscr{F} \text{ in } \mathbf{Sch}_{/S}}}{q_{\mathscr{F}} : X \ltimes \mathscr{F} \to X \text{ in } \mathbf{Sch}^{\mathrm{op}}_{/S}} \text{ Formal Dual}$$

so we can indeed use the projection π_0 as our bundle projection $q_{\mathscr{F}}$. The unit $\zeta_{\mathscr{F}}$ is similarly the formal dual of $\underline{\mathrm{Spec}}_X \iota_0$ for inclusion $\iota_0 : \mathcal{O}_X \to \mathcal{O}_X \ltimes \mathscr{F}$.

The Functor $\mathbf{QCoh}(X) \to \mathbf{DBun}(X)$ Part 2

Proposition

There is a functor $\mathbb{D} : \mathbf{QCoh}(X) \to \mathbf{DBun}_{\mathbf{Sch}^{\mathrm{op}}_{/S}}(X)$ given by

$$\mathscr{F} \mapsto \left(\begin{array}{cccc} X \ltimes \mathscr{F} & X & X \ltimes \mathscr{F}_2 & X \ltimes \mathscr{F} \\ \downarrow q_{\mathscr{F}} & \downarrow \zeta_{\mathscr{F}} & \downarrow \sigma_{\mathscr{F}} & \downarrow \lambda_{\mathscr{F}} \\ X & X \ltimes \mathscr{F} & X \ltimes \mathscr{F} & (X \ltimes \mathscr{F})[\varepsilon] \end{array} \right).$$

Remark

Because this talk is too long and defining $\sigma_{\mathscr{F}}$ and $\lambda_{\mathscr{F}}$ is technical, we have omitted it for now. If you are interested, please ask after or see the inevitable preprint!

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The Local Goal

We want to show that we can take a differential bundle in $\mathbf{Sch}^{\mathrm{op}}_{/S}$

$$\mathsf{q} = \begin{pmatrix} E & X & E_2 & E \\ \downarrow q, & \downarrow \zeta, & \downarrow \sigma, & \downarrow \lambda \\ X & E & E & E[\varepsilon] \end{pmatrix}$$

and turn it functorially into a quasi-coherent sheaf $\mathbb{K}\,\mathsf{q}$. This is much harder than constructing the functor $\mathbb{D}$.

Idea

Our goal, following [CruttwellLemay2023], is to show that $\mathbb{K}\,\mathsf{q}$ is the kernel of some map to $\mathcal{O}_X$. To build this map, we will need to know that every map in sight is affine!

Some Conventions

Definitions

For each map appearing in the differential bundle in $\mathbf{Sch}^{\mathrm{op}}_{/S}$,

$$\mathsf{q} = \begin{pmatrix} E & X & E_2 & E \\ \downarrow q, & \downarrow \zeta, & \downarrow \sigma, & \downarrow \lambda \\ X & E & E & E[\varepsilon] \end{pmatrix},$$

we will write a capital letter for the corresponding map in $\mathbf{Sch}_{/S}$. For instance, we have

$$\frac{\zeta : X \to E \text{ in } \mathbf{Sch}^{\mathrm{op}}_{/S}}{Z : E \to X \text{ in } \mathbf{Sch}_{/S}} \text{ Formal Dual}$$

The Easy-To-Show Affine Morphisms

Lemma

The morphisms $P : E \to E[\varepsilon]$ (the dual of the bundle projection $p_E : E[\varepsilon] \to E$) and $\Lambda : E[\varepsilon] \to E$ are affine morphisms.

Sketch.

Both P and Λ are naturally isomorphic to maps of the form $\tilde{P} : \underline{\mathrm{Spec}}_E(\mathcal{O}_E) \to \underline{\mathrm{Spec}}_E(\mathcal{O}_E[\varepsilon])$ and $\tilde{\Lambda} : \underline{\mathrm{Spec}}_E(\mathcal{O}_E[\varepsilon]) \to \underline{\mathrm{Spec}}_E(\mathcal{O}_E)$, respectively. □

The Section/Retraction Pair (Q, Z)

The Section/Retraction

By taking the formal dual of the identity $q \circ \zeta = \mathrm{id}_X$ in $\mathbf{Sch}^{\mathrm{op}}_{/S}$, we get that $Z \circ Q = \mathrm{id}_X$ holds in $\mathbf{Sch}_{/S}$. This means that, by using the sheaf map component of scheme morphisms, we have the section/retraction of sheaves of rings on X:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\quad Z^\sharp \quad} & Z_* \mathcal{O}_E \\ \| & & \downarrow{\scriptstyle Z_* Q^\sharp} \\ (Z \circ Q)_* \mathcal{O}_X & = & Z_*(Q_* \mathcal{O}_X) \end{array}$$

We need to show that Z is affine to deduce that this diagram takes place in $\mathbf{QCoh}(X)$.

The Map Q is Affine

Lemma

The morphism Q is a closed immersion and hence affine.

Sketch.

Differential bundles satisfy the identity $p_E \circ \lambda = \zeta \circ q$, so $\Lambda \circ P = Q \circ Z$. Because both Λ and P are affine, so too is $\Lambda \circ P = Q \circ Z$. From here some results from [EGA2] on the separation of affine morphisms allow us to deduce that Z is separated and that Q is a closed immersion; further results from [EGA2] give that Q is also affine. □

The Map Z is Affine and Consequences

Corollary

The morphism Z is affine.

Proof.

Use that affine morphisms are a pre-compositional ideal and the fact that both $\Lambda \circ P = Q \circ Z$ and Q are affine. □

Corollary

The sheaf $Z_* \mathcal{O}_E$ is quasi-coherent over X.

Why We Need Z Affine

Consequences of Z Being Affine

Because $\mathbf{QCoh}(X)$ is Abelian and $Z_* Q^\sharp$ is a split epimorphism, we have a split exact sequence in $\mathbf{QCoh}(X)$:

$$0 \longrightarrow \mathrm{Ker}(Z_* Q^\sharp) \xrightarrow{\ker(Z_* Q^\sharp)} Z_* \mathcal{O}_E \underset{Z^\sharp}{\overset{Z_* Q^\sharp}{\rightleftarrows}} \mathcal{O}_X \longrightarrow 0$$

Proposition

There is a functor $\mathbb{K} : \mathbf{DBun}(X) \to \mathbf{QCoh}(X)$ given by $\mathrm{q} \mapsto \mathrm{Ker}(Z_* Q^\sharp)$ on objects.

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The Current Goal

Our Task At Hand

Currently we must prove that for any scheme S and any S-scheme X, we have natural isomorphisms

$$\mathbb{K} \circ \mathbb{D} \cong \mathrm{id}_{\mathbf{QCoh}(X)}, \quad \mathbb{D} \circ \mathbb{K} \cong \mathrm{id}_{\mathbf{DBun}(X)}.$$

The former of these is much more straightforward, so we do that first.

The First Half of the Equivalence

Proposition

For any quasi-coherent sheaf $\mathscr{F}$, there is a natural isomorphism $(\mathbb{K} \circ \mathbb{D})(\mathscr{F}) \cong \mathscr{F}$.

Proof.

We make a direct computation/inspection that

$$(\mathbb{K} \circ \mathbb{D})(\mathscr{F}) = \mathsf{Ker}(\pi_0 : \mathcal{O}_X \ltimes \mathscr{F} \to \mathcal{O}_X) \cong \mathscr{F}\,.$$



Towards the Second Half of the Equivalence

What We Must Prove

Given a differential bundle in $\mathbf{Sch}^{\mathrm{op}}_{/S}$ of the form

$$\mathsf{q} = \begin{pmatrix} E & X & E_2 & E \\ \downarrow q\,, & \downarrow \zeta\,, & \downarrow \sigma\,, & \downarrow \lambda \\ X & E & E & E[\varepsilon] \end{pmatrix}$$

we compute that the fibre space of $\mathbb{D}(\mathbb{K}\,\mathsf{q})$ is the scheme $X \ltimes \mathrm{Ker}(Z_* Q^\sharp)$. As such we must prove that $E \cong X \ltimes \mathrm{Ker}(Z_* Q^\sharp)$ in a way suitably compatible with the lifts of each bundle.

The Second Half of the Equivalence

Proposition

There is a natural isomorphism of differential bundles $\mathsf{q} \cong (\mathbb{D} \circ \mathbb{K})(\mathsf{q})$.

Sketch.

Here is a step-by-step list on how to prove this.

1. First show that the maps $Z_* Q^\sharp : Z_* \mathcal{O}_E \to \mathcal{O}_X$ and $Z^\sharp : \mathcal{O}_X \to Z_* \mathcal{O}_E$, and $\hat{\lambda} : Z_* \mathcal{O}_E \to (Z_* \mathcal{O}_E)[\varepsilon]$ are the projection, unit, and lift of a differential bundle in the tangent category $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$.
2. Show that for every affine open subscheme $\gamma : U \to X$, evaluating every component of the bundle $Z_* \mathcal{O}_E$ at U gives a differential bundle in $\mathbf{CAlg}_{\mathcal{O}_X(U)}$.

[Continued] □

The Second Half of the Equivalence, Part 2

Sketch, Cont.

3. Use [CruttwellLemay2023] to give a linear bundle isomorphism $\alpha_U : (Z_* \mathcal{O}_E)(U) \to \mathcal{O}_X(U) \ltimes \mathrm{Ker}(Z_* Q_U^\sharp)$ in $\mathbf{CAlg}_{\mathcal{O}_X(U)}$ for all affine opens U.
4. Glue the α_U to an isomorphism $\hat{\alpha} : Z_* \mathcal{O}_E \to \mathcal{O}_X \ltimes \mathrm{Ker}(Z_* Q^\sharp)$ in $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$.
5. Use the formal dual of $\underline{\mathrm{Spec}}_X(\hat{\alpha})$ to give the desired isomorphism of bundles $\mathsf{q} \to (\mathbb{D} \circ \mathbb{K})(\mathsf{q})$.



Putting it All Together

Theorem

For any scheme S and any S-scheme X, there is an equivalence of categories

$$\mathbf{DBun}_{\mathbf{Sch}^{\mathrm{op}}_{/S}}(X) \simeq \mathbf{QCoh}(X).$$

Concluding Remarks

Corollary

For any scheme S and any S-scheme X, $\mathbf{DBun}_{\mathbf{Sch}^{\mathrm{op}}_{/S}}(\mathbb{1}) \simeq \{*\}$. That is, $\mathbf{Sch}^{\mathrm{op}}_{/S}$ has trivial differential objects.

Sketch.

The terminal object $\mathbb{1}$ in $\mathbf{Sch}^{\mathrm{op}}_{/S}$ is the initial object in $\mathbf{Sch}_{/S}$. Because $\operatorname{Spec} 0$ is initial in $\mathbf{Sch}_{/S}$, we see that $\mathbb{1} = \operatorname{Spec} 0$. Finally, as $|\operatorname{Spec} 0| = \varnothing$ we see that the only way there is a section/retraction $Q : E \to \mathbb{1}, Z : \mathbb{1} \to E$ is if $|E| = \varnothing$ as well. From here the result follows from the fact that $\mathbf{Shv}(\varnothing) \simeq \{*\}$. □

Concluding Remarks

An Observation

One of the themes in this talk is that the category $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$ is a pseudolimit tangent category (in the sense of [PronkVooys2025]) which mediates between Zariski descent, ring-theoretic techniques, and the less-than-linear aspects of scheme theory.
It would be interesting to see more work done directly with $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$ as a tangent category in its own right.

A General Pseudolimit Framework

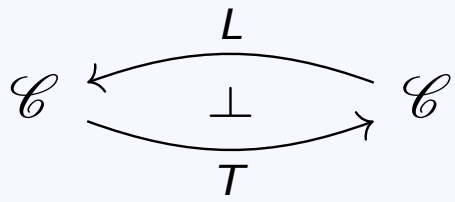
As we saw, much of the proof that $\mathbb{D} \circ \mathbb{K} \cong \mathrm{id}$ comes down to using the fact that $\mathbf{QCoh}(X, \mathbf{CAlg}_{\mathcal{O}_X})$ is a pseudolimit tangent category and that its differential bundles are themselves pseudolimits of differential bundles. It would be of interest to answer when, for a pseudofunctor $F : \mathscr{C}^{\mathrm{op}} \to \mathfrak{Tan}_{\mathbf{str}}$, we have an equivalence

$$\mathbf{DBun}\left(\mathrm{pslim}\, F\right) \overset{?}{\simeq} \mathrm{pslim}\, \mathbf{DBun}(F).$$

If this is satisfied for schemes, can we use this to streamline the given proof that $\mathbf{DBun}(X) \simeq \mathbf{QCoh}(X)$?

A General Framework (The Really Cool Framework)?

This is an example of what is likely a general phenomenon done the hard way. Assume there is an adjuction



in which $(\mathscr{C}, \mathbb{T})$ is a tangent category with T a right adjoint. As $(\mathscr{C}^{\mathrm{op}}, \mathbb{L})$ is then a tangent category with tangent functor L, it is natural to ask if there is an opposite equivalence

$$\mathbf{DBun}_{(\mathscr{C},\mathbb{T})}(X) \overset{?}{\simeq} \mathbf{DBun}_{(\mathscr{C}^{\mathrm{op}},\mathbb{L})}(X)^{\mathrm{op}}$$

for all $X : \mathscr{C}_0$? There is likely a tangent category theorist or two thinking about this, but I would love to know the answer!

References

Bibliography

- [CockettCruttwell2014] J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and SDG*, Appl. Categ. Structures 22 (2014), no. 2, 331 – 417.
- [CockettCruttwell2018] —, *Differential bundles and fibrations for tangent categories*, Cah. Topol. Géom. Différ. Catég. 59 (2018), no. 1, 10 -– 92.
- [CruttwellLemay2023] G. S. H. Cruttwell and J-S. P. Lemay, *Differential bundles in commutative algebra and algebraic geometry*, Theory Appl. Categ. 39 (2023), Paper No. 36, 1077 — 1120.
- [EGA2] A. Grothendieck and J. Dieudonné, *Élements de géometrie algébrique: II. étude globale élémentaire de quelques classes de morphismes,* Publications Mathématiques de l'IHES **32** (1967) 5 – 361.
- [LemayVooys2026] J.-S. P. Lemay and G. Vooys, *Important classes of morphisms and the relative cotangent sequence in tangetn categories*. Preprint avialable at arXiv:2506.07874.
- [MacAdam2021] B. MacAdam, *Vector bundles and differential bundles in the category of smooth manifolds*, Appl. Categ. Structures 29 (2021), no. 2, 285 — 310.

References

Bibliography, Cont.

- [Rosický1984] *Abstract tangent functors*, Diagrammes 12 (1984), JR1 — JR11.
- [PronkVooys2025] D. Pronk and G. Vooys, *Pseudolimits for tangent categories with applications to equivariant algebraic and differential geometry*, Math. Structures Comput. Sci., **35**, 2025, 80 pages.
- [Vooys2026] G. Vooys, *A Deep Dive Into the Tangent Category of Schemes*, preprint, 2026. Available at https://arxiv.org/abs/2608.06855.

The Last Slide

The End

Thanks for attending, listening, and giving me the opportunity to speak everybody! The preprint is on its way soon!